

- 1 Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$, such that

$$f(xf(y)) + f(f(x) + f(y)) = yf(x) + f(x + f(y))$$

holds for all $x, y \in \mathbb{R}$, where \mathbb{R} denotes the set of real numbers.

- 2 Suppose that we have $n \geq 3$ distinct colours. Let $f(n)$ be the greatest integer with the property that every side and every diagonal of a convex polygon with $f(n)$ vertices can be coloured with one of n colours in the following way:

- (i) At least two colours are used,
- (ii) any three vertices of the polygon determine either three segments of the same colour or of three different colours.

Show that $f(n) \leq (n - 1)^2$ with equality for infinitely many values of n .

- 3 Let $ABCD$ be a convex quadrilateral such that AB and CD are not parallel and $AB = CD$. The midpoints of the diagonals AC and BD are E and F , respectively. The line EF meets segments AB and CD at G and H , respectively. Show that $\angle AGH = \angle DHG$.

- 4 Determine all integers $k \geq 2$ such that for all pairs (m, n) of different positive integers not greater than k , the number $n^{n-1} - m^{m-1}$ is not divisible by k .

- 5 Let x, y, z be real numbers satisfying $x^2 + y^2 + z^2 + 9 = 4(x + y + z)$. Prove that

$$x^4 + y^4 + z^4 + 16(x^2 + y^2 + z^2) \geq 8(x^3 + y^3 + z^3) + 27$$

and determine when equality holds.

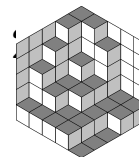
- 6 Let a, b, c be real numbers such that for every two of the equations

$$x^2 + ax + b = 0, \quad x^2 + bx + c = 0, \quad x^2 + cx + a = 0$$

there is exactly one real number satisfying both of them. Determine all possible values of $a^2 + b^2 + c^2$.

- 7 The numbers $0, 1, \dots, n$ ($n \geq 2$) are written on a blackboard. In each step we erase an integer which is the arithmetic mean of two different numbers which are still left on the blackboard. We make such steps until no further integer can be erased. Let $g(n)$ be the smallest possible number of integers left on the blackboard at the end. Find $g(n)$ for every n .

- 8 We colour every square of the 2009×2009 board with one of n colours (we do not have to use every colour). A colour is called connected if either there is only one square of that colour or any two squares of the colour can be reached from one another by a sequence of moves of a chess queen without intermediate stops at squares having another colour (a chess queen moves horizontally, vertically or diagonally). Find the maximum n , such that for every colouring of the board at least one colour present at the board is connected.



- 9] Let $ABCD$ be a parallelogram with $\angle BAD = 60$ and denote by E the intersection of its diagonals. The circumcircle of triangle ACD meets the line BA at $K \neq A$, the line BD at $P \neq D$ and the line BC at $L \neq C$. The line EP intersects the circumcircle of triangle CEL at points E and M . Prove that triangles KLM and CAP are congruent.
- 10] Suppose that $ABCD$ is a cyclic quadrilateral and $CD = DA$. Points E and F belong to the segments AB and BC respectively, and $\angle ADC = 2\angle EDF$. Segments DK and DM are height and median of triangle DEF , respectively. L is the point symmetric to K with respect to M . Prove that the lines DM and BL are parallel.
- 11] Find all pairs (m, n) of integers which satisfy the equation

$$(m + n)^4 = m^2n^2 + m^2 + n^2 + 6mn.$$

- 12] Find all non-negative integer solutions of the equation

$$2^x + 2009 = 3^y 5^z.$$