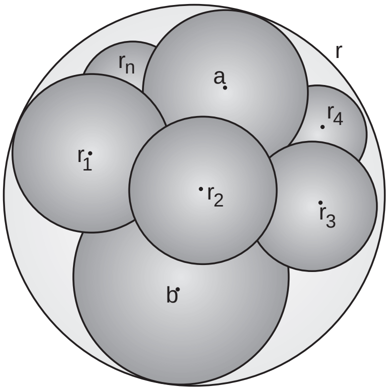

Figure 8.2. Six and only six spheres can be fitted in the "necklace" around spheres *a* and *b*.

we need say little more about it, except to repeat that it is more easily solved by inversion.

As for the Descartes circle theorem itself, this seems to have been the common property of traditional Japanese mathematicians. In the Descartes configuration it appeared on a *sangaku* in 1796, which was subsequently lost but recorded in the 1830 book *Saisi Sinzan*, or *Mathematical Tablets*, by Nakamura Tokikazu. For interested readers, we now give a Japanese proof from the 1830 *Sanpō Tenzan Syogakusyo*, or *Geometry and Algebra*, by Hasimoto Masakata. It is about at the level of the problems in chapter 5.

First, we need a preliminary result: Given three kissing circles with radii r_1, r_2, r_3, as in figure 8.3, show that

$$(AB)^2 = \frac{r_1^2 l^2}{(r_1 + r_2)(r_1 + r_3)}, \tag{1}$$

where l is the external tangent common to circles r_2 and r_3. This problem is an easy one and we leave it as an exercise for the reader.

Referring to figure 8.4 and noting that all the circles are mutually kissing, the preliminary result gives at once

$$(AB)^2 = \frac{r_3^2\left(2\sqrt{r_1 r_2}\right)^2}{(r_1 + r_3)(r_2 + r_3)} = \frac{4r_3^2 r_1 r_2}{(r_1 + r_3)(r_2 + r_3)},$$

$$(AC)^2 = \frac{4r_3^2 r_1 t}{(r_1 + r_3)(t + r_3)},\ (BC)^2 = \frac{4r_3^2 r_2 t}{(r_2 + r_3)(t + r_3)}. \tag{2}$$

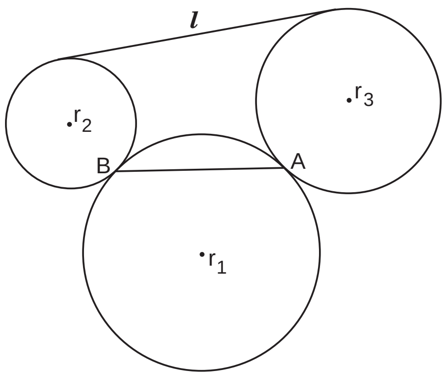

Figure 8.3. Find AB in terms of r_1, r_2, r_3, and l.

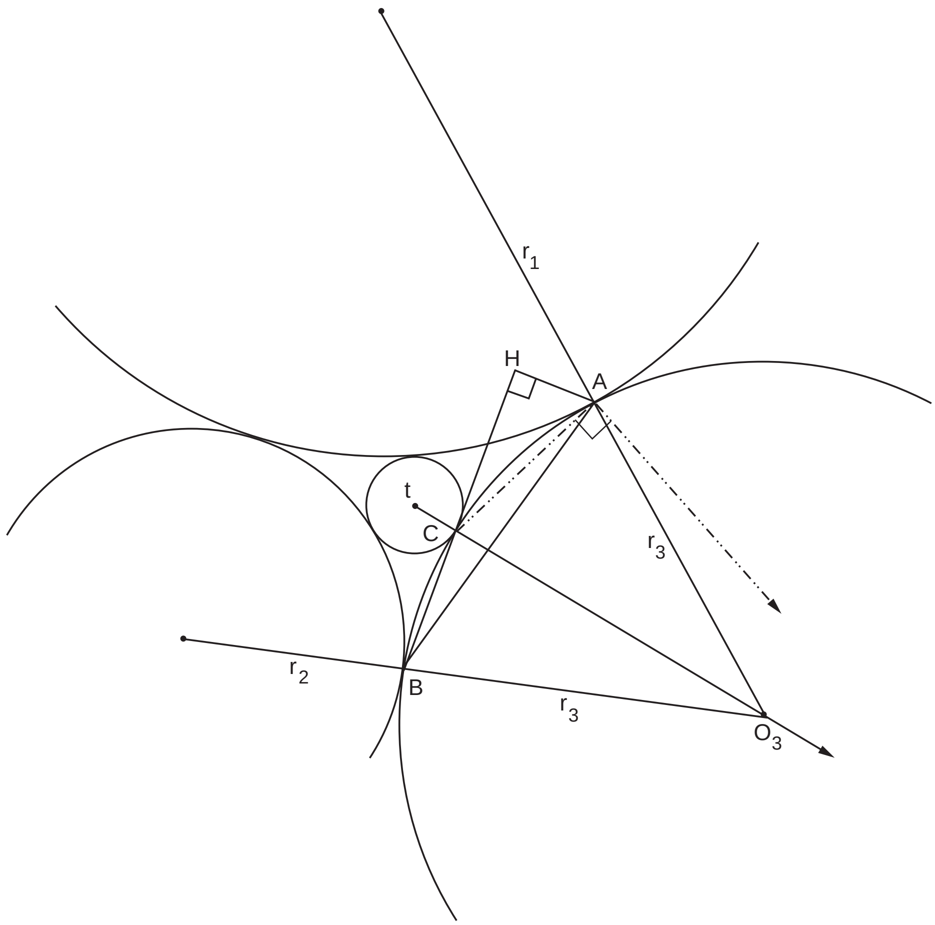

Figure 8.4. Because the indicated angle at A is a right angle, it is inscribed in a semicircle and the two arrows will intersect at the opposite end of the diameter containing CO_3 at a point C' (not shown).

The inscribed angles ABH and $AC'C$ subtend the same arc and so are equal. Thus, from similar right triangles,

$$\frac{AC}{2r_3} = \frac{AH}{AB},$$

which gives with equation (2)

$$(AH)^2 = \frac{(AB)^2(AC)^2}{4r_3^2} = \frac{4r_1^2 r_2 r_3^2 t}{(r_1+r_3)^2(r_2+r_3)(t+r_3)}.$$

From the Pythagorean theorem

$$(CH)^2 = (AC)^2 - (AH)^2 = \frac{4r_1 r_3^2 t}{(r_1+r_3)(t+r_3)}\left\{1 - \frac{r_1 r_2}{(r_1+r_2)(r_2+r_3)}\right\}. \tag{3}$$

Applying the law of cosines to ΔABC tells us that $(AB)^2 = (AC)^2 + (BC)^2 + 2(BC)(CH)$. Plugging in the values from equation (2) and (3) yields the ungainly

$$\frac{4r_3^2 r_1 r_2}{(r_1+r_3)(r_2+r_3)} = \frac{4r_3^2 r_1 t}{(r_1+r_3)(t+r_3)} + \frac{4r_3^2 r_2 t}{(r_2+r_3)(t+r_3)}$$
$$+ \; 2\frac{2r_3\sqrt{r_2 t}}{\sqrt{(r_2+r_3)(t+r_3)}}\,\frac{2r_3\sqrt{r_1 t}}{\sqrt{(r_1+r_3)(t+r_3)}}\sqrt{1 - \frac{r_1 r_2}{(r_1+r_2)(r_2+r_3)}},$$

which nonetheless readily simplifies to

$$r_1 r_2(t+r_3) = r_1 t(r_2+r_3) + r_2 t(r_1+r_3) + 2t\sqrt{r_1 r_2 r_3(r_1+r_2+r_3)},$$

or, solving for t,

$$t = \frac{r_1 r_2 r_3}{r_1 r_2 + r_1 r_3 + r_2 r_3 + 2\sqrt{r_1 r_2 r_3(r_1+r_2+r_3)}}.$$

We leave it to the reader to show that this is equivalent to result on page 285.

Closely related to the Descartes circle theorem and Soddy hexlet is the "Steiner chain" or "Steiner porism," after Jakob Steiner, who first considered such configurations in the West. Given two circles, one within the other but not concentric, we imagine trying to fit a chain of smaller circles of various sizes between them, each of which kisses both the inner and outer circles, as well as their nearest neighbors. We have already encountered similar configurations in chapter 6; the hexlet is a three-dimensional relative; a precise two-dimensional example is shown in plate 8.3.