Fuss' Problem of the Chord-Tangent Quadrilateral

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Nicolaus Fuss (1755-1826) was Swiss mathematician. He spent most of his life in St Petersburg in Russia, because he was recommended by Daniel Bernoulli for the post of Leonhard Euler's secretary. Working at the St. Petersburg Academy he wrote papers in spherical geometry, differential geometry, differential equations and many other topics.

Apart from other things N. Fuss investigated bicentric polygons, figures admitting both incircle and circumcircle, in other words both tangential and chordal. He found the relation between the radii and the line joining the centers of these circles for bicentric tetragon, pentagon, hexagon, heptagon and octagon.

Fuss' problem

To find the relation between the radii and the line joining the centers of the circles of circumscription and inscription of a bicentric quadrilateral.

It's useful to mention certain properties of chordal and tangential quadrilaterals before investigating bicentric quadrilateral.

Chordal or cyclic *quadrilateral* is a quadrilateral admitting circumcircle. The sum of its internal angles is the straight angle. Conversely, every convex quadrilateral with internal angles giving together the straight angle is chordal.

Tangential quadrilateral is a quadrilateral admitting incircle. Let a, b, c, d be the lengths of its sides clockwise or counterclockwise, then $a + c = b + d$ and conversely.

Bicentric quadrilateral

The tangency chords of the two pairs of opposite sides of a bicentric quadrilateral are perpendicular to each other (see figure 1).

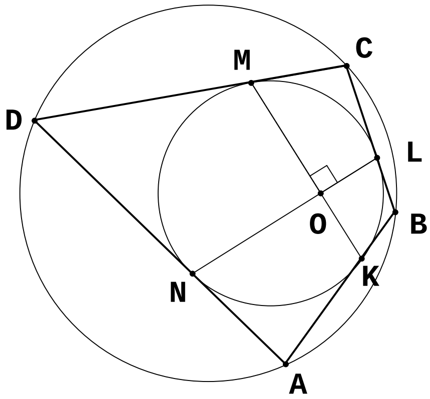


Figure 1: Bicentric quadrilateral

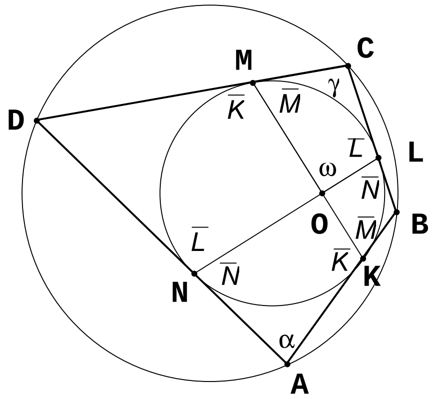


Figure 2: Designating the angles

Proof:

Let $ABCD$ be a bicentric quadrilateral; α and γ be the angles by the vertices A and C; $\bar{K}$, $\bar{L}$, $\bar{M}$ and $\bar{N}$ the angles by the tangent points K, L, M and N. The investigated angle formed by tangency chords is designated ω (see figure 2).

The lines AB and CD touch the incircle in points M and K, thus MK is the tangency chord and the angles by points M and K are congruent. The same stands for L and N. For quadrilaterals $AKON$ and $CMOL$:

$$\begin{aligned} \gamma + \bar{M} + \bar{L} + \omega &= 360^\circ \\ \alpha + \bar{K} + \bar{N} + \omega &= 360^\circ \end{aligned}$$

$$\underbrace{\alpha + \gamma}_{180^\circ} + \underbrace{\bar{M} + \bar{K}}_{180^\circ} + \underbrace{\bar{L} + \bar{N}}_{180^\circ} + 2\omega = 720^\circ \quad \Rightarrow \quad \omega = 90^\circ$$

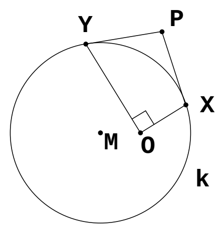
□

Conversely, every cyclic quadrilateral where the tangency chords of the two pairs of opposite sides are perpendicular to each other is bicentric. Consequently starting with two perpendicular chords of a circle, constructing tangents in their extremities, the bicentric quadrilateral is created.

The locus

Lines KM and LN divide the bicentric quadrilateral $ABCD$ in four quadrilaterals $AKON$, $BLOK$, $CMOL$ and $DNOM$ with some common properties. Examining this sort of quadrilateral $OXPY$ by investigating locus of points P is conductive:

- Let k be a circle, O be the point inside,
- X, Y be the points on k giving right angle XOY,
- P be the point of intersection of tangents touching k in X and Y.



Visualizing the locus (see figure 3) the locus seems to be a circle.

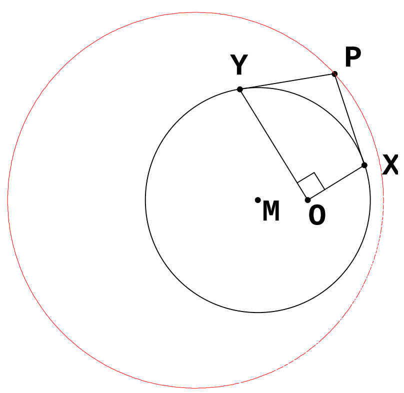


Figure 3: Visualization of the locus

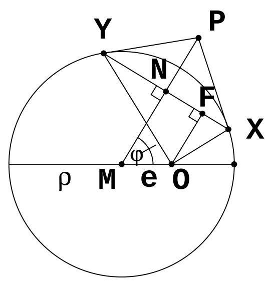


Figure 4: Designating of situation

Proof:

Designating points see figure 4. Let e be the length of MO, φ the size of the angle OMP, ρ the radius of k and p the length of MP. In the right-angled triangle OXY $|OF|^2 = |FX| \cdot |FY|$. The line MP is the bisector of XY, thus $|NX| = |NY|$ and the angle MNY is right. Furthermore $|NF| = e{\cdot}\sin\varphi$. Therefore

$$|OF| = |MN| - e \cdot \cos\varphi$$

$$|FX| = |NX| - e \cdot \sin\varphi$$

$$|FY| = |NX| + e \cdot \sin\varphi$$

Substituting foregoing into $|OF|^2 = |FX| \cdot |FY|$

$$\begin{aligned} (|MN| \; - e \cdot \cos\varphi)^2 &= (|NX| - e \cdot \sin\varphi)(|NX| + e \cdot \sin\varphi) \\ |MN|^2 - 2\,|MN|\, e \cdot \cos\varphi + \; e^2\cos^2\varphi &= |NX|^2 - e^2 \cdot \sin^2\varphi \\ |MN|^2 - 2\,|MN|\, e \cdot \cos\varphi + \; e^2 &= |NX|^2 \end{aligned}$$

As $|MX| = \rho$, then in the right-angled triangle MXN $|NX|^2 = \rho^2 - |MN|^2$. Then

$$2\,|MN|^2 - 2\,|MN|\, e \cdot \cos\varphi + \; e^2 \; = \; \rho^2.$$

In right-angled triangle MXP is $|MX|^2 = |MP| \cdot |MN|$ or $\rho^2 = p\,|MN|$. Substituting for $|MN|$ and arranging:

$$\frac{2\rho^4}{\rho^2 - \; e^2} \; = \; 2\frac{\rho^2 e}{\rho^2 - \; e^2} p\cos\varphi + \; p^2.$$

p and φ are variables dependent on the position of point P; ρ and e are invariable for given circle k and point O. Assuming P lying on a circle with the radius $r = |SP|$ and the centre S lying on MO, let $x = |SM|$ be the distance of centers. Then in the triangle SMP there is $r^2 = \; x^2 + \; p^2 + 2xp\cos\varphi$.

Comparing last two equations, r is constant for $x = \; \frac{\rho^2 e}{\rho^2 - e^2}$. Hence the desired locus is a circle with the centre S and radius r. Substituting for x and eliminating e we get the relation between the radii of given circle ρ, found circle r and the distance of their centers x:

$$2\rho^2(r^2 + \; x^2) = \; (r^2 - \; x^2)^2.$$

□

Conclusion

Given two circles, one inside another. Starting in P, drawing a tangent to the inner circle, from the point of intersection with the outer circle a tangent to inner one again and so on, we return to P and get the bicentric quadrilateral. The position of P on the outer circle is arbitrary, moving P we obtain all bicentric quadrilaterals for given incircle and circumcircle. Foregoing procedure leads not only to proving the locus is a circle but above all to the solution of the Fuss' problem:

Given ρ and r the radii of incircle and circumcircle, then the distance x of their centers satisfies the equation

$$2\rho^2(r^2 + \; x^2) = \; (r^2 - \; x^2)^2.$$

For $x \in (0, r - \rho)$, $r \geq \rho\sqrt{2}$ this equation has one solution

$$x = \sqrt{r^2 + \rho^2 - \rho\sqrt{4r^2 + \rho^2}}.$$

For $r < \rho\sqrt{2}$ it has no solution and no bicentric quadrilateral can be constructed. Derived equation can be arranged:

$$\frac{1}{(r-x)^2} + \frac{1}{(r+x)^2} = \frac{1}{\rho^2}.$$

References

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