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ON A THEOREM OF SYLVESTER AND SCHUR

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In 1892, Sylvester [7] proved that in the set of integers $n, n+1, \dots, n+k-1$, $n > k > 1$, there is a number containing a prime divisor greater than k . This theorem was rediscovered, in 1929, by Schur [6]. More recent results include an elementary proof by Erdős [1] and a proof of the following theorem by Faulkner [2]: Let p_k be the least prime $\geq 2k$; if $n \geq p_k$ then $\binom{n}{k}$ has a prime divisor $\geq p_k$ with the exceptions $\binom{9}{2}$ and $\binom{10}{3}$. In that paper the author uses some deep results of Rosser and Schoenfeld [5] on the distribution of primes. A note by Moser [4] states that a simple extension of Erdős' proof leads to the result that the product of k consecutive integers greater than k is divisible by a prime $\geq \frac{1}{10}k$.

The object of this note is to prove by elementary means the following theorem:

THEOREM. *The product of k consecutive integers $n(n+1) \cdots (n+k-1)$ greater than k contains a prime divisor greater than $\frac{2}{3}k$ with the exceptions 3.4, 8.9 and 6.7.8.9.10.*

We may reformulate the theorem as follows: If $n \geq 2k$ then $\binom{n}{k}$ contains a prime divisor greater than $\frac{2}{3}k$ with the above exceptions.

COROLLARY. *For all $k > 1, n \geq 2k, \binom{n}{k}$ has a prime divisor $\geq \frac{7}{5}k$.*

The result of the corollary is suggested in [4].

The first part of the following proof employs methods similar to those used by Erdős in [1]. In [3] we proved by elementary means the following: The product of the prime powers less than or equal to n is less than 3^n for $n > 1$, i.e. if $\alpha = \alpha(p, n)$ is such that $p^\alpha \leq n < p^{\alpha+1}$, then $\prod_{p \leq n} p^\alpha < 3^n$. It is this result that enables us to extend Erdős' work.

Since the exponent β_p to which a prime occurs in $\binom{n}{k}$ is

$$\beta_p = \sum_{i=1}^{\lfloor \log_p n \rfloor} \left(\left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{n-k}{p^i} \right\rfloor - \left\lfloor \frac{k}{p^i} \right\rfloor \right)$$

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it is easy to see that

LEMMA 1. *If $p^{\beta v} \parallel \binom{n}{k}$ then $p^{\beta v} \leq n$.*

Proof of the theorem. (1). Let $\pi(k)$ denote the number of primes $\leq k$. Clearly for $k \geq 8$, $\pi(k) \leq \frac{1}{2}k$. Thus if $\binom{n}{k}$ has no prime factor greater than $\frac{3}{2}k$, Lemma 1 implies

$$\binom{n}{k} \leq n^{1/2 \cdot 3/2k} \leq n^{3/4k}.$$

However since

$$\binom{n}{k} = \frac{n}{k} \dots \frac{n-1}{k-1} \dots \frac{n-k+1}{1} > \left(\frac{n}{k}\right)^k$$

we must have

$$\left(\frac{n}{k}\right)^k < n^{3/4k}$$

which is false if $k \leq n^{1/4}$. Therefore our theorem holds for $8 \leq k \leq n^{1/4}$.

It is easy to see that $\pi(k) < \frac{1}{3}k$ for $k \geq 37$ and $\pi(k) < \frac{2}{3}k$ for $k \leq 300$. In a similar manner as above we then have that the theorem is true for $37 < k \leq n^{1/2}$ and $300 \leq k \leq n^{2/3}$ in these cases respectively.

(2). We now consider the case $k > n^{2/3}$. If $\binom{n}{k}$ contains no prime divisor exceeding $\frac{3}{2}k$ then by Lemma 1

$$(1) \quad \binom{n}{k} < \prod_{p \leq 3/2k} p \prod_{p \leq n^{1/2}} p \prod_{p \leq n^{1/3}} p \dots$$

In [3] we proved by elementary methods that

$$(2) \quad 3^{n_0} > \prod_{p \leq n_0} p \prod_{p \leq n_0^{1/2}} p \prod_{p \leq n_0^{1/3}} p \dots$$

Therefore, since $k > n^{2/3}$ implies $k^{1/l} > n^{1/(2l-1)}$ for $l \geq 2$, we have

$$(3) \quad 3^{3/2k} > \prod_{p \leq 3/2k} p \prod_{p \leq n^{1/3}} p \prod_{p \leq n^{1/5}} p \dots$$

Now taking $n_0 = n^{1/2}$ in (2), we find

$$(4) \quad 3^{n^{1/2}} > \prod_{p \leq n^{1/2}} p \prod_{p \leq n^{1/4}} p \prod_{p \leq n^{1/6}} p \dots$$

Combining (1), (3) and (4) we have under the assumption that $\binom{n}{k}$ is not divisible by any prime exceeding $\frac{3}{2}k$ that

$$(5) \quad \binom{n}{k} < 3^{3/2k+n^{1/2}}$$

It is easy to prove by induction that $\binom{4k}{k} > \left(\frac{4^4}{3^3}\right) \frac{1}{4k}$. Assume that $n \geq 4k$. Then (5) implies

$$(6) \quad 3^{3/2k+n^{1/2}} > \left(\frac{4^4}{3^3}\right)^k \frac{1}{4k}.$$

It now follows from (6) that

$$\left(\frac{3}{2}k + n^{1/2}\right) \log 3 > k(4 \log 4 - 3 \log 3) - \log 4k$$

and under the initial assumption that $k > n^{2/3}$ that

$$n^{1/2} \log 3 > n^{2/3}(8 \log 2 - \frac{9}{2} \log 3) - \log n$$

which is false if $n > 240$.

We now assume $3k \leq n < 4k$. Inductively we can show $\binom{3k}{k} > \left(\frac{3^3}{2^2}\right)^k \frac{1}{3k}$, then as above we have

$$3^{3/2k+n^{1/2}} > \left(\frac{3k}{k}\right) > \left(\frac{3^3}{2^2}\right)^k \frac{1}{3k}$$

which implies

$$\left(\frac{3}{2}k + n^{1/2}\right) \log 3 > k(3 \log 3 - 2 \log 2) - \log 3k.$$

But since $n < 4k$, we have

$$2k^{1/2} \log 3 > k\left(\frac{3}{2} \log 3 - 2 \log 2\right) - \log 3k,$$

which is false for $k > 120$ and our theorem holds for $n \geq 480$.

It now only remains to check the cases where $2k \leq n < 3k$, $k > n^{2/3}$. We first prove the following.

LEMMA 2. *There is a prime between $3n$ and $4n$ for $n > 1$.*

Proof. Assume the contrary. Consider the binomial coefficient $\binom{4n}{n}$. It is easy to see that no prime p , such that $2n < p \leq 3n$ divides $\binom{4n}{n}$. Thus our assumption is that no prime between $2n$ and $4n$ occurs in $\binom{4n}{n}$.

If α_p is the exponent of p in $\binom{4n}{n}$ then

$$\alpha_p = \sum_{i=1}^{\lceil \log_p 4n \rceil} \left(\left[\frac{4n}{p^i} \right] - \left[\frac{3n}{p^i} \right] - \left[\frac{n}{p^i} \right] \right).$$

Since each term appearing in this sum is either 0 or 1 for any p , if $\alpha_p \geq 2$ then $p \leq (4n)^{1/2}$. It now follows that under our assumption

$$(7) \quad \binom{4n}{n} < \prod_{p^{\alpha_p} \leq 2n} p \prod_{p \leq (4n)^{1/2}} p$$

since if $p^{\alpha} \leq 2n < p^{\alpha+1}$ then $4n < p^{\alpha+2}$. On the other hand we can prove by induction that $\binom{4n}{n} > \left(\frac{4^4}{3^3}\right) \frac{n_1}{4n}$. By (2) and (7) we then have

$$\left(\frac{4^4}{3^3}\right) \frac{1}{4n} < 3^{2n+(4n)1/2}$$

which is false for $n \geq 2200$, and a straight-forward check of a table of primes for $1 \leq n < 2200$ concludes the proof of Lemma 2.

If we now consider the case $2k \leq n < 3k, k > n^{2/3}$, our conclusion holds for $k > 4$ by Lemma 2 since there is a prime between $[\frac{2}{3}n]$ and n , and $[\frac{2}{3}n] \geq \frac{2}{3}k$.

Thus our theorem holds for $k \geq 8$ with a finite number of exceptions which may be checked by a table of primes.

(3) Consider the case $k=5$, we want to show that $n(n-1) \cdots (n-4)$ where $n-4 > 5$ is divisible by a prime ≥ 11 . Assume the contrary and consider the binomial coefficient $\binom{n}{5}$. By Lemma 1 we have

$$\frac{n(n-1) \cdots (n-4)}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \binom{n}{5} < n^{\pi((3/2)5)} = n^4$$

which is certainly false for say $n \geq 129$. A check of tables of primes for $n \leq 129$ reveals one exception to our theorem i.e. 6.7.8.9.10 has no prime divisor > 7 . We may treat the case $k=4$ in the same manner and no exceptions occur.

The cases $k=6$ and $k=7$ now follows from the case $k=5$ since $\frac{3}{2} \cdot 6 < \frac{3}{2} \cdot 7 < 11$ and the product of any five consecutive numbers greater than 6 contains a prime divisor ≥ 11 .

For $k=3$, consider the integers $n, n+1, n+2, n > 3$. If $n \equiv 0 \pmod{3}$, then either n or $n+1$ is divisible by a prime greater than 3 since $(n, n+1)=1$ and $n > 3$. The case $n+2 \equiv 0 \pmod{3}$ is identical. If $n+1 \equiv 0 \pmod{3}$ the only time whether neither n or $n+2$ is divisible by a prime greater than 3 is when n and $n+2$ are powers of 2 i.e. when $n=2$. Therefore our theorem holds for $k=3$.

When $k=2$, by the same approach we only have the exceptions 3.4 and 8.9, since the only solutions to $2^\alpha - 3^\beta = \pm 1$ are $\alpha=2, \beta=1$ and $\alpha=3, \beta=2$. The case $k=1$ is trivially true.

The exception $\binom{10}{5}$ proves the corollary to the theorem i.e. that $\frac{7}{5}$ is the “best possible” constant c such that $\binom{n}{k}$ is divisible by a prime $\geq ck$ for $n \geq 2k$.

REFERENCES

1. P. Erdős, *A theorem of Sylvester and Schur*, J. London Math. Soc. **9** (1934), 282–288.
2. M. Faulkner, *On a theorem of Sylvester and Schur*, J. London Math. Soc. **41** (1966), 107–110.
3. D. Hanson, *On the product of the primes*, Canad. Math. Bull., (1) **15** (1972), 33–37.

4. L. Moser, *Insolvability of $\binom{2n}{n} = \binom{2a}{a}\binom{2b}{b}$* , *Canad. Math. Bull. (2)* **6** (1963), 167–169.
5. J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, *Illinois J. Math.* **6** (1962), 64–94.
6. I. Schur, *Einige Satze uber Primzahlen mit wendung auf Irreduzibilitatsfragen*, *Sitzungberichte der preussischen Akedemie der Wissenschaften, Phys. Math. Klasse*, **23** (1929), 1–24.
7. J. J. Sylvester, *On arithmetical series*, *Messenger of Mathematics*, **XXI** (1892), 1–19, 87–120, and *Mathematical Papers*, **4** (1912), 687–731.

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