Solution for Crux 5005

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1 Problem

Let $x, y, z > 0$ such that $xyz = 1$. Show that

$$\left(\frac{x}{1+x+xy}+\frac{y}{1+y+yz}+\frac{z}{1+z+zx}\right)^3 \leq \frac{x^3}{1+x+xy}+\frac{y^3}{1+y+yz}+\frac{z^3}{1+z+zx}$$

Proof:
We know from Hölder's Inequality that

$$\left(\sum_{cyclic}\frac{x}{1+x+xy}\right)^3 \leq \left(\sum_{cyclic}\frac{x^3}{1+x+xy}\right)\left(\sum_{cyclic}\frac{1}{1+x+xy}\right)^2$$

Hence, it sufficies to show that

$$\sum_{cyclic}\frac{1}{1+x+xy} \leq 1$$

Recall that $xyz = 1$. Thus,

$$\sum_{cyclic}\frac{1}{1+x+xy} = \frac{1}{1+x+xy}+\frac{1}{1+y+yz}+\frac{1}{1+z+zx}$$

$$= \frac{1}{1+x+xy}+\frac{1}{1+y+yz}+\frac{1}{xyz+z+zx} = \frac{1}{1+y+yz}+\left(\frac{z+1}{z}\right)\frac{1}{1+x+xy}$$

$$= \frac{1}{1+y+yz}+\left(\frac{z+1}{xz}\right)\frac{1}{1+y+yz} = \frac{1+z+zx}{xz(1+y+yz)} = \frac{1+z+zx}{1+z+x} = 1$$

as desired. The equality case satisfies (Because of Hölder)

$$\frac{x^3}{(1+x+xy)^2} = \frac{y^3}{\left(1+y+\frac{1}{x}\right)^2} = \frac{x^2y^3}{(1+x+xy)^2}$$

hence equality occurs for $x = y = z = 1$.