# Contest Inequalities- Generalizations

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## 1 International Mathematical Olympiad- IMO

**Generalization 1.1**
Let $x_1, x_2, \cdots, x_n, r$ be positive reals ($n \geq 3$) such that $\prod x_1 = \lambda$. Then prove that

$$\sum_{j=1}^{n} \frac{\sqrt[n-1]{x_j^{r(2n-1)}} - x_j^r}{\sqrt[n-1]{x_j^{r(2n-1)}} + \sum_{k=1}^{n} (x_k^r) - x_j^r} \geq \frac{1}{\lambda} - 1$$

*Proof.*

Generalization 1 creates way too many similar problems with original IMO 2005 3 problem. For instance

By giving $r = n - 1$ under same conditions, we get a new problem

**Generalization 1.0**
Let $x_1, x_2, \cdots, x_n$ be positive reals ($n \geq 3$) such that $\prod x_1 = \lambda$. Then prove that

$$\sum_{j=1}^{n} \frac{x_j^{2n-1} - x_j^{n-1}}{x_j^{2n-1} + \sum_{k=1}^{n} \left(x_k^{n-1}\right) - x_j^{n-1}} \geq \frac{1}{\lambda} - 1$$

**Application on IMO 2005 Problem 3**

$x, y, z$ be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0$$

*Proof.*
The problem is a special case of Generalization 1.1 where

$$n = 3, r = 2, \lambda = 1$$

**Generalization 2**
Let $a, b, c, p, \lambda$ be positive reals. Then prove that

$$\frac{a^p}{\sqrt{a^{2p} + \lambda bc}} + \frac{b^p}{\sqrt{b^{2p} + \lambda ca}} + \frac{c^p}{\sqrt{c^{2p} + \lambda ab}} \geq \frac{\lambda^2}{4\left(3\sqrt[3]{\lambda^2} + 4\right)}$$

*Proof.*

**Application on IMO 2001 Problem 2**
Prove that for all positive real numbers $a, b, c$

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1$$

*Proof.*
The original problem is a special case of Generalization 2 where

$$p = 1, \lambda = 8$$

**Generalization 3**
Let $a, b, c, d$ be reals $(a \geq b \geq c \geq d > 0)$ such that $\alpha \geq \theta \geq \beta \geq \lambda > 0$ and

$$6\lambda \geq \alpha + \beta \geq 2\theta$$

inequalities hold. Then prove that

$$(\lambda a + \beta b + \theta c + \alpha d)\, a^a b^b c^c d^d < \lambda$$

*Proof.*

**Application on IMO 2020 Problem 2**
The real numbers $a, b, c, d$ are such that $a \geq b \geq c \geq d > 0$ and $a+b+c+d = 1$.
Prove that

$$(a + 2b + 3c + 4d)a^a b^b c^c d^d < 1$$

*Proof.*
The problem is a special case of Generalization 3 where

$$\lambda = 1, \beta = 2, \theta = 3, \alpha = 4$$

and the given inequality

$$6\alpha \geq \alpha + \beta \geq 2\theta$$

holds.

**Generalization 4.1**
Let $a, b, c, n$ be positive reals $(n \geq 3)$ such that $abc = k^3$. Then prove that

$$\frac{1}{a^n (b+c)} + \frac{1}{b^n (c+a)} + \frac{1}{c^n (a+b)} \geq \frac{3}{2.k^{n+1}}$$

*Proof.*

**Generalization 4.2**
Let $a_1, a_2, \cdots, a_n, n, p$ be positive reals $(n \geq 2\ , p \geq 1)$ such that $\prod a_1 = 1$. Then prove that

$$\sum_{cyc}\left(\frac{1}{a_1^{2p+1}\left(x_1.\dfrac{\prod a_1}{a_1 a_n} + x_2.\dfrac{\prod a_1}{a_1 a_2}\right)^p}\right) \geq \frac{n}{(x_1 + x_2)^p}$$

*Proof.*

**Generalization 4.3**
Let $a_1, a_2, \cdots, a_n, n, p, k$ be positive reals $(n \geq 2\ , p \geq 1)$ such that $\prod a_1 = k$. Then prove that

$$\sum_{cyc}\left(\frac{k^{p+1}}{a_1^{2p+1}\left(x_1.\dfrac{\prod a_1}{a_1 a_n} + x_2.\dfrac{\prod a_1}{a_1 a_2}\right)^p}\right) \geq \frac{n\sqrt[n]{k^{n-1}}}{(x_1 + x_2)^p}$$

*Proof.*

**Application on IMO 1995 Problem 2**
Let $a, b, c$ be positive reals such that $abc = 1$. Then prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

*Proof.*
The problem is a special case of Generalization 4.1 where

$$n = 2, k = 1$$

And also a special case of Generalization 3 where

$$n = 3, p = 1, k = 1, x_1 = 1, x_2 = 1$$

**Generalization 5**
Let $x_1, x_2, \ldots, x_{2p+1}$ be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1^y + x_2^y + \cdots + x_n^y)\left(\frac{1}{x_1^k} + \frac{1}{x_2^k} + \cdots + \frac{1}{x_n^k}\right)}$$

is an integer for $n = 1, 2, \cdots, 2p+1$. Then prove that

$$a_{2p+1} \geq x_1^{y-k} + p\left(1 + 2\sqrt[4]{(x_{n-1}x_n)^{y-k}}\right)$$

holds.

*Proof.*

**Application on IMO 2023 Problem 4**
Let $x_1, x_2, \ldots, x_{2023}$ be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \ldots, 2023$. Prove that $a_{2023} \geq 3034$.

*Proof.*

The problem is a special case of Generalization 5 where

$$y = 1, k = 1, p = 1011$$

**Generalization 6**
Let $a, b, c, \lambda$ be positive reals ($\lambda \leq 1$) such that $abc = 1$. Then prove that

$$\left(a - \lambda + \frac{\lambda}{b}\right)\left(b - \lambda + \frac{\lambda}{c}\right)\left(c - \lambda + \frac{\lambda}{a}\right) \leq \lambda\sqrt{\lambda \prod \left(\frac{a + 1 - \lambda}{c} + \left(\sqrt{\lambda} - 1\right)^2\right)}$$

*Proof.*

**Application on IMO 2000 Problem 2**
Let $a, b, c$ be positive real numbers so that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1$$

*Proof.*
The problem is a special case of Generalization 6 where

$$\lambda = 1$$

with a finishing $\sqrt{\prod \frac{a}{c}} = 1$.

**Generalization 7**
$x, y, z, \lambda$ reels ($\lambda \neq 0$) each different from $\sqrt[3]{\lambda^2}$ hold $xyz = \lambda^2$. Then prove that

$$\sum_{cyc}\left(\frac{x}{x - \sqrt[3]{\lambda^2}}\right)^2 \geq \lambda^2 + 2\sqrt[6]{\lambda^5}\left(1 - \sqrt{\lambda}\right)\left(\sum_{cyc}\frac{x}{\sqrt[3]{\lambda^2}} + 1\right)$$

*Proof.*

**Application on IMO 2008 Problem 2**
(a) Prove that

$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2}\geq 1$$

for all real numbers $x$, $y$, $z$, each different from 1, and satisfying $xyz = 1$.

*Proof.*
The problem is a special case of Generalization 7 where

$$\lambda = 1$$

## 2 International Mathematical Olympiad- IMO Shortlist

**Generalization 8**
Let $a, b, c, \theta, \lambda$ be positive reals such that $ab + bc + ca = \beta$ and

$$\beta^4\left(3\theta+\lambda\beta\right)=3^{5-n}$$

equalities hold. Then prove that

$$\sqrt[n]{\frac{\theta}{a}+\lambda b}+\sqrt[n]{\frac{\theta}{b}+\lambda c}+\sqrt[n]{\frac{\theta}{c}+\lambda a}\leq\frac{1}{\sqrt[3]{(abc)^n}}$$

*Proof.*

**Application on IMO Shortlist 2004 Problem A.5**
Let $a, b, c$ be positive reals such that $ab + bc + ca = 1$. Then prove that

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}$$

*Proof.*
The problem is a special version of Generalization 8 where

$$\beta = 1, \theta = 1, \lambda = 6, n = 3$$

with the condition hold.

**Generalization 9**
Let $a_1, a_2, \cdots, a_{2^p-x}, \lambda$ be positive reals $(1 \leq x+1 \leq 2^p)$ such that $\min(a_i a_j) \geq 1$. Then prove that

$$\sqrt[2^p-x]{\prod a_1^2 + \lambda} \leq \left(\frac{\sum_{cyc} a_1}{2^p - \lambda}\right)^2 + \lambda$$

*Proof.*

**Application on IMO Shortlist 2016 Problem A.1**
Let $a$, $b$, $c$ be positive real numbers such that $\min(ab, bc, ca) \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

*Proof.*

The problem is a special case of Generalization 9 where

$$p = 2, x = 1, \lambda = 1$$

**Generalization 10.1**
Let $a, b, c$ be positive reals such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \lambda (a + b + c)$. Then prove that

$$\frac{1}{((\lambda+\beta)\,a+\lambda b+\beta c)^2}+\frac{1}{((\lambda+\beta)\,b+\lambda c+\beta a)^2}+\frac{1}{((\lambda+\beta)\,c+\lambda a+\beta b)^2}\leq\frac{3}{16\beta}$$

*Proof.*

**Generalization 10.2**
Let $a, b, c$ be positive reals such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \theta (a + b + c)$. Then prove that

$$\frac{1}{((\lambda+\beta)\,a+\lambda b+\beta c)^2}+\frac{1}{((\lambda+\beta)\,b+\lambda c+\beta a)^2}+\frac{1}{((\lambda+\beta)\,c+\lambda a+\beta b)^2}\leq\frac{3\theta}{16\lambda\beta}$$

*Proof.*

**Application on IMO Shortlist 2009 Problem A.2**
Let $a$, $b$, $c$ be positive real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = a + b + c$. Prove that

$$\frac{1}{(2a+b+c)^2}+\frac{1}{(a+2b+c)^2}+\frac{1}{(a+b+2c)^2}\leq\frac{3}{16}.$$

*Proof.*

By giving

$$\alpha = 1, \beta = 1, \theta = 1$$

in Generalization 10.2 leads us to the maximum value $\frac{3}{16}$.

**Generalization 11.1**
Let $a_1, a_2, \cdots, a_n$ be positive reals such that $\sum_{cyc-j} a_j a_{j+1} \cdots a_{j-2} \leq n \prod a_1$. Then prove that

$$\sum_{cyc-k} \sqrt{\frac{a_k^2 + a_{k+1}^2}{a+b}} + n \leq \sqrt{2} \left( \sum_{cyc-i} \sqrt{a_i + a_{i+1}} \right)$$

*Proof.*

**Generalization 11.2**
Let $a_1, a_2, \cdots, a_n$ be positive reals such that $\sum_{cyc-j} a_j a_{j+1} \cdots a_{j-2} \leq \theta \prod a_1$. Then prove that

$$\sum_{cyc-k} \sqrt{\frac{a_k^2 + a_{k+1}^2}{a+b}} + n\sqrt{\frac{n}{\theta}} \leq \sqrt{2} \left( \sum_{cyc-i} \sqrt{a_i + a_{i+1}} \right)$$

*Proof.*

**Application on IMO Shortlist 2009 Problem A.4**
Let $a$, $b$, $c$ be positive real numbers such that $ab + bc + ca \leq 3abc$. Prove that

$$\sqrt{\frac{a^2+b^2}{a+b}} + \sqrt{\frac{b^2+c^2}{b+c}} + \sqrt{\frac{c^2+a^2}{c+a}} + 3 \leq \sqrt{2} \left( \sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a} \right)$$

*Proof.*
This problem is a special case of Generalization 11.2 where

$$n = 3, \theta = 3$$

**Generalization 12**
Let $a_1, a_2, \cdots, a_{2p}$ be positive reals ($p \geq 2$) such that

$$(a_1 + a_3 + \cdots + a_{2p-1})(a_2 + a_4 + \cdots + a_{2p}) = \sqrt[p]{(a_1 a_3 \cdots a_{2p-1})^2} + \sqrt[p]{(a_2 a_4 \cdots a_{2p})^2}$$

holds. Then prove that

$$\sum_{cyc-j} \frac{a_j}{a_{j+1}} \geq p^3$$

*Proof.*

**Application on IMO Shortlist 2020 Problem A.3**
$a, b, c, d$ are positive real numbers satisfying $(a + c)(b + d) = ac + bd$. Find the smallest possible value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$$

*Proof.*
The problem is a special case or Generalization 12 where

$$p = 2$$

which gives us the minimum value 8.

**Generalization 13**
Let $a_1, a_2, \cdots, a_n$ be positive reals such that $\prod a_1 = 1$ and

$$\sum_{cyc} a_1 > \sum_{cyc-j} \frac{a_j}{a_{j+1}}$$

expressions hold. Then prove that

$$\sum_{cyc} a_1 < \sum_{cyc-j} \frac{a_j^2 a_{j+1} a_{j+2}}{\prod a_1}$$

*Proof.*

**Application on IMO Shortlist 2008 A.5**
Let $a, b, c, d$ be positive reals satysfying $abcd = 1$ and

$$a + b + c + d < \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$$

expressions. Then prove that

$$a + b + c + d < \frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{a}{d}$$

*Proof.*
The problem is a special case of Generalization 13 where

$$n = 2$$

**Generalization 14**
Let $a, b, c$ be positive reals such that $abc = k$. Then prove that

$$\sum_{cyc} \frac{(ab)^n}{k\left(ab\right)^n + a^{2n+3} + b^{2n+3}} \leq \frac{1}{k}$$

*Proof.*

**Application on IMO Shortlist 1996 Problem A.1**

Suppose that $a, b, c > 0$ such that $abc = 1$. Prove that

$$\frac{ab}{ab + a^5 + b^5} + \frac{bc}{bc + b^5 + c^5} + \frac{ca}{ca + c^5 + a^5} \leq 1.$$

*Proof.*

The problem is a special case of Generalization 14 where

$$n = 1, k = 1$$

**Generalization 15**

x$_0, x_1, ..., x_n, \lambda, \theta_1$ positive reals $(n \geq 1)$ holds $0 = x_0 < x_1 < x_2 < \cdots < x_n$. Additionally $\theta_1, \theta_2, \cdots, \theta_n$ is a nondecreasing arithmetic sequence. Then prove that for $a$ reals satisfying the following

$$\frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1} + \cdots + \frac{1}{x_n - x_{n-1}} \geq a\left(\frac{\left(\lambda - \theta_1\right)\left(\lambda + \theta_1\right)}{2x_1\left(\lambda + \theta_1 - \theta_2\right)} + \sum_{i=2}^{n} \frac{\lambda + \theta_{i-1} + \theta_i}{2x_i}\right)$$

we have

$$a \leq \frac{2\left(\lambda + \theta_1 - \theta_2\right)}{\lambda^2}$$

$\|$*Proof.*

**Application on IMO Shortlist 2016 Problem A.8**
$Findthelargestrealconstantasuchthatforall$n $\geq 1$ and for all real numbers $x_0, x_1, \cdots, x_n$ satisfying $0 = x_0 < x_1 < x_2 < \cdots < x_n$ we have

$$\frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1} + \cdots + \frac{1}{x_n - x_{n-1}} \geq a\left(\frac{2}{x_1} + \frac{3}{x_2} + \cdots + \frac{n+1}{x_n}\right)$$

*Proof.*
The problem is a special case of Generalization 15 where

$$\theta_1 = 1, \theta_2 -_{\theta} {}_1 = 1, \lambda = 3$$

with a maximum $\dfrac{4}{9}$

# 3 International Mathematical Olympiad- IMO Longlist

**Generalization 16**
Let $a_1, a_2, \cdots, a_n$ be positive real numbers greater than with order $\lambda_1, \lambda_2, \cdots, \lambda_n$ and $\sum_{i=1}^{n} \dfrac{\lambda_i}{x_i} = n - 1$. Then prove that

$$\sqrt{\sum_{cyc} x_1} \geq \sum_{j=1}^{n} x_j - \lambda_j$$

*Proof.*

**Application on IMO Longlist 1992 Problem 21**
Prove that if $x, y, z > 1$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$, then

$$\sqrt{x + y + z} \geq \sqrt{x - 1} + \sqrt{y - 1} + \sqrt{z - 1}.$$

*Proof.*
The problem is a special case of Generalization 15 where

$$n = 3, \lambda_1 = \lambda_2 = \lambda_3 = 1$$

# 4 Asia Pasific Mathematical Olympiad- APMO

**Generalization 17.1**
Let $a_1, a_2, \cdots, a_n$ $(n \geq 2)$ be positive reals. Then prove that

$$\prod_{i=1}^{n} (a_i + 2) \geq \frac{3^{n-2}}{2^{n-3}} \left( \sum_{cyc} a_1 \right)^2$$

*Proof.*

**Generalization 17.2**
Let $a_1, a_2, \cdots, a_n, k$ $(n \geq 2)$ be positive reals. Then prove that

$$\prod_{i=1}^{n} \left(a_i^2 + k\right) \geq \frac{3^{n-2}.2^4}{k^{n+1}} \left( \sum_{cyc} a_1 \right)^2$$

*Proof.*

**Application on APMO 2004 Problem 5**
Prove that the inequality

$$\left(a^2+2\right)\left(b^2+2\right)\left(c^2+2\right) \geq 9\left(ab+bc+ca\right)$$

holds for all positive reals $a, b, c$.

*Proof.*
By Generalization 16.1 and 16.2, we know that

$$LHS \geq 3\left(\sum_{cyc} a_1\right) \geq 9\left(ab+bc+ca\right)$$

which completes the proof.

We can say that Generalization 16.2 proves the stronger inequality of APMO 2004 5.

**Generalization 18**
Let $a, b, c$ positive reals. Then prove that

$$\left(\lambda+\frac{a^k}{b^p}\right)\left(\lambda+\frac{b^k}{c^p}\right)\left(\lambda+\frac{c^k}{a^p}\right) \geq \left(\frac{3\sum_{cyc}\left(a^k\sqrt{b^{k-p}}\right)-\sum_{cyc}a^p}{\sqrt{(abc)^p}}+\lambda^2+\frac{(abc)^{k-p}}{\lambda}\right)$$

*Proof.*

**Application on APMO 1998 Problem 3**
Let $a, b, c$ be positive reals. Then prove that

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \geq 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$$

*Proof.*

The problem is a special case of Generalization 17 where

$$\lambda = 1, k = 1, p = 1$$

**Generalization 19**
Let $x, y, z$ be positive reals. Then prove that

$$\sum_{cyc} \left( \frac{x^{k+p} + (yz)^p}{\sqrt[n]{\lambda . x^{np} (y+z)}} \right) \geq \frac{\sqrt[n]{\frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^{nk-1}}{2.3^{nk-2}}} + \sqrt[n]{\frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^{np-1}}{2}}}{\sqrt[n]{\lambda}}$$

*Proof.*

**Application on APMO 2007 Problem 4**
Let $x; y$ and $z$ be positive real numbers such that $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$. Prove that

$$\frac{x^2 + yz}{\sqrt{2x^2(y+z)}} + \frac{y^2 + zx}{\sqrt{2y^2(z+x)}} + \frac{z^2 + xy}{\sqrt{2z^2(x+y)}} \geq 1$$

*Proof.*
The problem is a special case of Generalization 18 where

$$\sum_{cyc} n = 3, \lambda = 8, \sqrt{x_1} = 1, p = 1, k = 1$$

with a minimum 1.

# 5 Balkan Mathematical Olympiad

**Generalization 20**
Let $a_1, a_2, \cdots, a_{3n}$ be positive reals ($n \geq 1$). Then prove that

$$\sum_{cyc}\left(\frac{1}{a_1a_2\cdots a_n\left(a_{n+1}a_{n+2}\cdots a_{2n}+1\right)}\right)\geq\frac{3n}{\prod a_1+1}$$

*Proof.*

**Application on Balkan MO 2006 Problem 1**
Let $a, b, c$ be positive real numbers. Prove the inequality

$$\frac{1}{a\left(b+1\right)}+\frac{1}{b\left(c+1\right)}+\frac{1}{c\left(a+1\right)}\geq\frac{3}{1+abc}$$

*Proof.*
The problem is a special case of Generalization 19 where

$$n=1$$

**Generalization 21**
Let $a_1, a_2, \cdots, a_n$ be positive reals $(n\geq 3)$. Then prove that

$$\sum_{cyc}\left(\frac{\prod\left(a_1\right)\frac{a_k}{a_{k-1}}\left(a_{k+1}-a_{k-1}\right)}{a_k+a_{k+1}}\right)\geq 0$$

*Proof.*

**Application on Balkan MO 2010 Problem 1**
Let $a, b, c$ be positive real numbers. Prove that

$$\frac{a^2b(b-c)}{a+b}+\frac{b^2c(c-a)}{b+c}+\frac{c^2a(a-b)}{c+a}\geq 0$$

*Proof.*
The problem is a special version of Generalization 20 where

$$n = 3$$

**Generalization 22**
For $a_1, a_2, \cdots, a_p, k \in \mathbf{R}^+$ such that

$$\sum_{i=1}^{p} \frac{a_1 a_2 \cdots a_n}{a_i} = k(a_1 a_2 \cdots a_p)$$

Prove the following

$$a_1^{n+1}a_2^n + a_2^{n+1}a_3^n + \cdots + a_{p-1}^{n+1}a_p^n + a_p^{n+1}a_1^n \geq (n+1)(a_1 + a_2 + \cdots + a_p) - kn$$

Determine the equality case.

*Proof.*

$$\sum_{i=1 \to p} \frac{a_1 a_2 \cdots a_n}{a_i} = k(a_1 a_2 \cdots a_p) \Rightarrow \sum_{cyc} \frac{1}{a_1} = k$$

By homogenising $k$ with the given expression

$$\sum_{cyc} a_1^{n+1}a_2^n \geq (n+1)(a_1 + a_2 + \cdots + a_p) - kn$$

$$\Rightarrow \sum_{cyc} a_1^{n+1}a_2^n + n\left(\sum_{cyc} \frac{1}{a_1}\right) \geq (n+1)(a_1 + a_2 + \cdots + a_p)$$

$$= \sum_{cyc} \left( a_1^{n+1} a_2^n + \overbrace{\frac{1}{a_2} + \frac{1}{a_2} + \cdots + \frac{1}{a_2}}^{n} \right) \overbrace{\geq}^{AM-GM} \sum_{cyc} (n+1)a_1 = (n+1)(a_1 + a_2 + \cdots + a_n)$$

Equality case occurs when

$$a_1 = a_2 = \cdots = a_p = 1$$

Proof has been completed.

**Application on Balkan MO 2014 Problem 3**
Let $x, y$ and $z$ be positive real numbers such that $xy + yz + xz = 3xyz$. Prove that

$$x^2y + y^2z + z^2x \geq 2(x + y + z) - 3$$

and determine when equality holds.

*Proof.*
This problem is a special case of Generalization 21 where $n = 1$ and $k = 3$. That's why we get

$$x^2y + y^2z + z^2x \geq (n+1)(a_1 + a_2 + \cdots + a_p) - kn = 2(x + y + z) - 3$$

# 6 Balkan Mathematical Olympiad Shortlist

**Generalization 23.1**
Let $a, b, c$ be positive reals such that $abc = k$. Then prove that

$$\frac{1}{k\left(a^{k+4}+b^{k+4}\right)+c^2}+\frac{1}{k\left(b^5+c^5\right)+a^2}+\frac{1}{k\left(c^5+a^5\right)+b^2}\leq\frac{1}{\sqrt[3]{k+8}}$$

*Proof.*

**Generalization 23.2**
Let $a_1, a_2, \cdots, a_n, n, k$ be positive reals such that $\prod a_1 = k$. Then prove that

$$\sum_{cyc}\frac{1}{k\left(\sum_{i=1}^{n-1}a_i^{n+k+1}\right)+a_n^{k+1}}\leq\frac{1}{\sqrt[n]{k+2n+2}}$$

*Proof.*

**Application on Balkan MO Shortlist 2017 Problem A.1**
Let $a, b, c$ be positive reals such that $abc = 1$. Then prove that

$$\frac{1}{a^5+b^5+c^2}+\frac{1}{b^5+c^5+a^2}+\frac{1}{c^5+b^5+b^2}\leq 1$$

*Proof.*
The problem is a special case of Generalization 22.2 where

$$n = 3, k = 1$$

**Generalization 24.1** Let $a_1, a_2, a_3, n$ $(n \geq 2)$ be positive reels such that $a_1a_2a_3 = 1$. Then prove that

$$k\left(a_1^k+a_2^k+a_3^k\right)\left(\frac{1}{a_1^k}+\frac{1}{a_2^k}+\frac{1}{a_3^k}\right)\geq 3\left(a_1^{k-1}+a_2^{k-1}+a_3^{k-1}+\sum_{p=1,i=k-1}^{p=3,i=1}\left(a_p^ia_{p+1}^{k-i}\right)\right)$$

*Proof.*

**Generalization 24.2**
Let $a_1, a_2, \cdots, a_n, n, k$ be positive reels $(n, k \geq 2)$ such that $\prod a_1 = 1$. Then prove that

$$k\left(\sum_{cyc} a_1^k\right)\left(\sum_{cyc}\frac{1}{a_1^k}\right) \geq n\left(\sum_{cyc} a_1^{k-1} + \sum_{p=1,i=k-1}^{p=n,i=1}\left(a_p^i a_{p+1}^{n-i}\right)\right)$$

*Proof.*

**Application on Balkan MO Shortlist 2018 Problem A.4**
Let $a, b, c$ be positive real numbers such that $abc = 1$ Prove that

$$2(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \geq 3\left(a+b+c+ab+bc+ca\right)$$

*Proof.*
The problem is a special case of Generalization 23.2 where

$$n = 2, k = 3$$

**Generalization 25.1**
Let $a, b, c, \lambda.\theta, \beta, \phi, p$ positive reals such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 3$ and

$$4\phi\beta = 3\left(\lambda+\theta+2\right)$$

expressions hold. Then prove that

$$\frac{\sqrt{\lambda a+\frac{\theta b}{c}}+\sqrt{\lambda b+\frac{\theta c}{a}}+\sqrt{\lambda c+\frac{\theta a}{b}}}{\beta}\leq\frac{\phi\left(a+b+c\right)-p}{\sqrt{2}}$$

*Proof.*

**Generalization 25.2**
Let $a,b,c,\lambda.\theta,\beta,\phi,p$ positive reals such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\alpha$ and

$$\frac{2p\beta+3\left(\theta+2\right)}{9}=\frac{2\phi\beta-\lambda}{\alpha}$$

expressions hold. Then prove that

$$\frac{\sqrt{\lambda a+\frac{\theta b}{c}}+\sqrt{\lambda b+\frac{\theta c}{a}}+\sqrt{\lambda c+\frac{\theta a}{b}}}{\beta}\leq\frac{\phi\left(a+b+c\right)-p}{\sqrt{2}}$$

*Proof.*

**Application on Balkan MO Shortlist Problem A.2**
Let $a,b,c$ be positive reals such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$. Then prove that

$$\frac{\sqrt{a+\frac{b}{c}}+\sqrt{b+\frac{c}{a}}+\sqrt{c+\frac{a}{b}}}{3}\leq\frac{a+b+c-1}{\sqrt{2}}$$

*Proof.*
The problem is a special version of Generalization 25.2 where the given equation holds and

$$\alpha=3,\beta=3,\phi=1,\theta=1,\lambda=1,p=1$$

## 7 Junior Balkan Mathematical Olympiad

**Generalization 26**
Let $a_1, a_2, \cdots, a_{2k+1}$ be positive reals. Then prove that

$$\sum_{n=1}^{2k+1} \frac{1}{a_n\left(a_{n-2k+1}+a_{n-2k+2}+\cdots+a_n\right)} \geq \frac{(2k+1)^3}{2k\left(\sum_{cyc} a_k\right)^2}$$

*Proof.*
Let us form the sum into cyclic inequality

$$\sum_{cyc} \frac{1}{a_{2k}(a_1+a_2+\cdots+a_{2k})} \geq p$$

$$\Rightarrow S=\left(\sum_{cyc} \frac{1}{a_{2k}(a_1+a_2+\cdots+a_{2k})}\right)\left(\sum_{cyc} a_1\right)\left(\sum_{cyc}(a_1+a_2+\cdots+a_{2k})\right)$$

$$\geq p\left(\sum_{cyc} a_1\right)\left(\sum_{cyc}(a_1+a_2+\cdots+a_{2k})\right)$$

Let us work with left hand side

*i)*

$$\left(\sum_{cyc} \frac{1}{a_{2k}(a_1+a_2+\cdots+a_{2k})}\right) \overbrace{\geq}^{AM-GM} (2k+1) . \sqrt[2k+1]{\frac{1}{\left(\prod a_1\right)\left(\prod a_1+a_2+\cdots+a_{2k}\right)}}$$

*ii)*

$$\left(\sum_{cyc} a_1\right) \overbrace{\geq}^{AM-GM} (2k+1).\sqrt[2k+1]{\prod a_1}$$

*iii)*

$$\left(\sum_{cyc} (a_1+a_2+\cdots+a_{2k})\right) \overbrace{\geq}^{AM-GM} (2k+1).\sqrt[2k+1]{\prod a_1+a_2+\cdots+a_{2k}}$$

Then

$$S \geq (2k+1)^3.\sqrt[2k+1]{\left(\frac{1}{\left(\prod a_1\right)\left(\prod a_1+a_2+\cdots+a_{2k}\right)}\right)\left(\prod a_1\right)\left(\prod a_1+a_2+\cdots+a_{2k}\right)}$$

$$= (2k+1)^3$$

$$S \geq (2k+1)^3 \geq p\left(\sum_{cyc} a_1\right)\left(\sum_{cyc} (a_1+a_2+\cdots+a_{2k})\right)$$

$$\Rightarrow p \leq \frac{(2k+1)^3}{\left(\sum_{cyc} a_1\right)\left(\sum_{cyc} (a_1+a_2+\cdots+a_{2k})\right)} = \frac{(2k+1)^3}{\left(\sum_{cyc} a_1\right).2k\left(\sum_{cyc} a_1\right)} = \frac{(2k+1)^3}{2k\left(\sum_{cyc} a_1\right)^2}$$

Lastly

$$\sum_{cyc} \frac{1}{a_{2k}(a_1+a_2+\cdots+a_{2k})} \geq \frac{(2k+1)^3}{2k\left(\sum_{cyc} a_1\right)^2} \geq p$$

Proof completed.

**Application on JBMO 2002 Problem 4**

Prove that for all positive real numbers $a, b, c$ the following inequality takes place

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2}$$

*Proof.*
Substituting $k = 1$ in Generalization 1, we get as desired.

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{(2k+1)^3}{2k\left(\sum_{cyc} a_k\right)^2} = \frac{27}{2(a+b+c)^2}$$

**Generalization 27.1**
Let $a, b$ be positive reals and $\lambda$ be integer $(\lambda \geq 1)$ such that $ab \geq 1$. Then prove that

$$\left(a + \lambda b + \frac{\lambda}{a+1}\right)\left(b + \lambda a + \frac{\lambda}{b+1}\right) \geq \left(\frac{(\lambda+1)(\lambda+2) - 4}{\lambda}\right)^2$$

*Proof.*

**Generalization 27.2**
Let $a, b, k$ be positive reals and $\lambda$ be integer $(\lambda \geq 1)$ such that $ab \geq k$. Then prove that

$$\left(a + \lambda b + \frac{\lambda}{a+k}\right)\left(b + \lambda a + \frac{\lambda}{b+k}\right) \geq \left(\frac{(\lambda+1)(\lambda+2)}{\lambda}\right)^2 \cdot \sqrt[\lambda^2+3\lambda-2]{k^{(\lambda^2-1)}}$$

*Proof.*

**Generalization 27.3**
Let $a, b, k$ be positive reals and $\lambda, \phi$ be integers $(\lambda, \phi \geq 1)$ such that $ab \geq k$.

Then prove that

$$\left(\phi a+\lambda b+\frac{\lambda}{a+k}\right)\left(\phi b+\lambda a+\frac{\lambda}{b+k}\right)\geq\left(\lambda^2+\lambda\left(\phi+2\right)-2\right)^2\left(\sqrt[\lambda^2+\lambda(\phi+2)-2]{k^{(\lambda^2+\lambda(\phi-1)-1)}}\right)$$

*Proof.*

**Application on JBMO 2013 Problem 3**
Show that

$$\left(a+2b+\frac{2}{a+1}\right)\left(b+2a+\frac{2}{b+1}\right)\geq 16$$

for all positive real numbers $a$ and $b$ such that $ab\geq 1$

*Proof.*
The problem is a special case of Generalization 23.3 where

$$\phi=1,\lambda=2,k=1$$

**Generalization 28.1**
Let $x_1, x_2, \cdots, n$ $(n\geq 2)$ be nonnegative reals (Each not equal to 0). Then prove that

$$\sum_{cyc}\frac{(n-1)x_1^2-(n-2)x_1+\sum_{2\leq i\leq n}x_i}{x_1+\sum_{2\leq i\leq n}x_i^2}\geq n$$

and determine when does the equality holds?

*Proof.*

**Generalization 28.2**

Let $x_1, x_2, \cdots, n, p, k$ $(n \geq 2)$ be nonnegative reals (Not all equal to 0). Then prove that

$$\sum_{cyc} \frac{(n-1)x_1^p - (n-2)x_1^k + \sum_{2\leq i\leq n} x_i^k}{x_1^k + \sum_{2\leq i\leq n} x_i^p} \geq n$$

and determine when does the equality holds?

*Proof.*

**Application on JBMO 2023 Problem 2**
Prove that for all non-negative real numbers $x, y, z$, not all equal to 0, the following inequality holds

$$\frac{2x^2 - x + y + z}{x + y^2 + z^2} + \frac{2y^2 + x - y + z}{x^2 + y + z^2} + \frac{2z^2 + x + y - z}{x^2 + y^2 + z} \geq 3$$

Determine all the triples $(x, y, z)$ for which the equality holds.

*Proof.*
The problem is a special case of Generalization 24.2 where

$$n = 3, p = 2, k = 1$$

**Generalization 29**
Prove that for $\lambda \in \left(2, \dfrac{13 + 3\sqrt{21}}{2}\right)$ reals, the $(a, b)$ positive integer pairs which holds the inequality

$$\lambda ab \leq a^3 - b^3 \leq (\lambda + 1)\, ab$$

are

$$b \in \left( \frac{(\lambda-2)^3}{27\,(3b+\lambda-2)}, \frac{(\lambda-2)^3}{18\,(3b+\lambda-2)} \right)$$

for $b$ between below is

$$(a,b) = \left( \frac{3b+\lambda-2}{3}, b \right)$$

*Proof.*

This generalization creates different problems with different criterions. For example, our original JBMO 2022 Problem 1 has one solution without equality however by giving $\lambda = 14$ we get a new problem with an equality case $(a,b) = (8,4)$.

**Version 29.1**
Find all pairs of positive integer $(a,b)$ that satysfies the inequality

$$14ab \leq a^3 - b^3 \leq 15ab$$

*Proof.*

**Application on JBMO 2022 Problem 1**
Find all pairs of positive integers $(a,b)$ such that

$$11ab \leq a^3 - b^3 \leq 12ab$$

*Proof.*
By Generalization 25 we can say that there is one solution which is

$$\left( \frac{3b+\lambda-2}{3}, b \right) = (5,2)$$

because of there is just one $b$ inside of the given values 2. Adapting Generalization 25 to the problem is left to the reader.

**Generalization 30**
For positive real numbers $a_1, a_2, \cdots, a_n$ $(n > 2)$ with $\prod a_1 = (n-2)^n$ . Then prove that

$$\sum_{cyc}\left(a_1 + \frac{n-2}{a_2}\right)^2 \geq n\left(\sum_{cyc} a_1 + 1\right)$$

*Proof.*

$$\sum_{cyc}\left(a_1 + \frac{n-2}{a_2}\right)^2 \overbrace{\geq}^{Titu} \frac{\left(\sum_{cyc} a_1 + k(\sum_{cyc} \frac{1}{a_1})\right)^2}{n} \overbrace{\geq}^{AM-GM} \frac{\left(\sum_{cyc} a_1 + n\right)^2}{n}$$

$$\frac{\left(\sum_{cyc} a_1 + n\right)^2}{n} \geq n\left(\sum_{cyc} a_1 + 1\right)$$

Let us prove the last expression

By denoting $\sum_{cyc} a_1 = k$

$$(k+n)^2 \geq n^2(k+1)$$
$$k^2 + 2nk + n^2 \geq n^2k + n^2 \rightarrow k^2 + 2nk \geq n^2k$$
$$k + 2n \geq n^2$$

Last inequality is true because of

$$k = \sum_{cyc} a_1 \geq n\sqrt[n]{\prod a_1} = n\sqrt[n]{(n-2)^n} = n(n-2)$$

Problem has been completed.

**Application on JBMO 2014 Problem 3**
For positive real numbers $a, b, c$ with $abc = 1$ prove that

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \geq 3(a+b+c+1)$$

*Proof.*
Placing $n = 3$ in Generalization 2, we get the following.

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \geq n\left(\sum_{cyc} a_1+1\right) = 3(a+b+c+1)$$

Problem has been solved.

## 8 Junior Balkan Mathematical Olympiad Short-list

**Generalization 31**
Let $x, y, z$ be positive reals such that $xy + yz + zx = p$. Then prove that

$$\frac{x^k+\lambda}{y+z}+\frac{y^k+\lambda}{z+x}+\frac{z^k+\lambda}{x+y}+p \geq \frac{\sqrt{(3p)^k}\left(\lambda.3^k+p\sqrt{p}\left(\sqrt{3^kp^{k-3}}+2\sqrt{3^{2k-3}}\right)\right)\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}{2.3^{k-1}\left(x+y+z\right)^{k+2}}$$

*Proof.*

**Application on JBMO 2022 Shortlist Problem A.2**
Let $x, y$, and $z$ be positive real numbers such that $xy + yz + zx = 3$. Prove that

$$\frac{x+3}{y+z}+\frac{y+3}{z+x}+\frac{z+3}{x+y}+3 \geq 27 \cdot \frac{(\sqrt{x}+\sqrt{y}+\sqrt{z})^2}{(x+y+z)^3}$$

*Proof.*
The problem is a special case of Generalization 29 where

$$\lambda=3, p=3, k=1$$

**Generalization 32**
Let $a, b, c, p, k$ be positive reals such that $a+b+c=1$. Then prove that

$$a. \sqrt[k]{\frac{b^p}{a^\lambda}}+b. \sqrt[k]{\frac{c^p}{b^\lambda}}+c. \sqrt[k]{\frac{a^p}{c^\lambda}} \leq \frac{3\,(ab+bc+ca)}{k}+\frac{2\lambda-2p-1}{k}+1$$

*Proof.*

**Application on JBMO Shortlist 2022 Problem A.3**
Let $a, b$, and $c$ be positive real numbers such that $a+b+c=1$. Prove the following inequality

$$a \sqrt[3]{\frac{b}{a}}+b \sqrt[3]{\frac{c}{b}}+c \sqrt[3]{\frac{a}{c}} \leq ab+bc+ca+\frac{2}{3}$$

*Proof.*
The problem is a special case of Generalization 30 where

$$k=3, p=1, \lambda=1$$

**Generalization 33**

Let $a_1, a_2, \cdots, a_n$ be positive reals ($n \geq 3$) such that $\sum_{cyc} a_1 \geq \sum_{cyc} \frac{1}{a_1}$. Then prove that

$$\sum_{cyc-k} \left( \frac{\frac{\prod a_1}{a_{k+1}a_k} + \frac{\prod a_1}{a_k a_{k-1}} - \frac{\prod a_1}{a_{k+1}a_{k+2}}}{\frac{\prod a_1}{a_k} \left(\sum_{cyc} a_1\right)} \right) \leq 1$$

*Proof.*

**Application on JBMO Shortlist 2022 Problem A.4**
Suppose that $a, b$, and $c$ are positive real numbers such that

$$a + b + c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

Find the largest possible value of the expression

$$\frac{a+b-c}{a^3+b^3+abc} + \frac{b+c-a}{b^3+c^3+abc} + \frac{c+a-b}{c^3+a^3+abc}$$

*Proof.*
The problem is a special version of Generalization 31 where

$$n = 3$$

with the maximum 1.

# 9 ELMO Shortlist

**Generalization 34**
Let $a_1, a_2, \cdots, a_n$ be positive reals such that $\sum_{cyc} \frac{1}{a_1} = 1$. Then prove that

$$\prod a_1 \left(\sum_{cyc} a_1^{a_1-1}\right) \geq n^n \cdot \sum_{j=1}^{n} \left(\frac{\prod a_1}{a_j}\right)$$

*Proof.*

**Application on ELMO Shortlist 2019 Problem A.1**
Let $a, b, c$ be positive reals such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Then prove that

$$a^a bc + b^b ca + c^c ab \geq 27ab + 27bc + 27ca$$

*Proof.*
The problem is a special case of Generalization 32 where

$$n = 3$$

# 10 Centroamerican Mathematical Olympiad

**Generalization 35**
Let $a, b, c$ be positive reals such that $a + b + c = 1$. Then prove that

$$a\sqrt{pa^2 + kbc} + b\sqrt{pb^2 + kac} + c\sqrt{pc^2 + kab} \leq \frac{k\sqrt{\frac{k}{2} - p}}{2\,(k - 2p)}$$

*Proof.*

**Application on Centroamerican 2019 Problem 5**
Let $a, b, c$ be positive reals such that $a+b+c=1$. Then prove that

$$a\sqrt{a^2+6bc}+b\sqrt{b^2+6ac}+c\sqrt{c^2+6ab}\leq\frac{3\sqrt{2}}{4}$$

*Proof.*
The problem is a special version of Generalization 33 where

$$p=1, k=6$$

# 11 Turkey Team Selection Test

**Generalization 36.1**
Let $x, y, z, \lambda, a_1, b_1$ be positive reals. Then prove that

$$\lambda\left(xy+yz+xz\right)+\frac{a_1\left(b_1+1\right)}{b_1x}+\frac{b_1\left(a_1+1\right)}{a_1y}+\frac{a_1+b_1}{a_1b_1z}\geq 3\sqrt[3]{\lambda a_1b_1}\left(\frac{a_1+b_1+a_1b_1}{a_1b_1}\right)$$

*Proof.*

**Generalization 36.2**
Let $x_1, x_2, \cdots, x_n, \lambda, a_1, a_2, \cdots, a_{n-1}$ be positive reals $(n\geq 3)$. Then prove that

$$\sum_{j=1}^{n-1}\left(\frac{a_j\left(1+\sum_{p=1}^{n-1}\left(\frac{1}{a_p}\right)-\frac{1}{a_j}\right)}{x_j}\right)+\frac{\sum_{k=1}^{n-1}\left(\frac{1}{a_k}\right)}{x_n}\geq n\sqrt[n]{\lambda\prod a_1}\left(1+\sum_{p=1}^{n-1}\frac{1}{a_p}\right)$$

*Proof.*

**Application on Turkey TST 2022 Problem 7**
Let $x, y, z$ be positive reals. Then find minimum value of

$$xy + yz + zx + \frac{1}{x} + \frac{2}{y} + \frac{5}{z}$$

*Proof.*
The problem is a special version of Generalization 34.2 where

$$n = 3, \lambda = 1, a_1 = \frac{1}{3}, b_1 = \frac{1}{2}$$

with minimum $3\sqrt[3]{36}$

**Generalization 37.1**
Let $a, b, c, k \in \mathbf{R}^+$ and $k \geq 1$ such that $a + b + c = 3$. Then prove the following

$$a^k b + b^k c + c^k a + 3k \geq (k+1)(ab + bc + ca)$$

*Proof.*

$$a^k b + b^k c + c^k a + 3k = a^k b + b^k c + c^k a + k(a + b + c)$$

$$= a^k b + (k-1)b + b^k c + (k-1)c + c^k a + (k-1)a + (a + b + c)$$

$$= a^k b + \overbrace{b + \cdots + b}^{k-1} + b^k c + \overbrace{c + \cdots + c}^{k-1} + c^k a + \overbrace{a + \cdots + a}^{k-1} + (a + b + c)$$

$$\overbrace{\geq}^{AM-GM} k(ab+bc+ca)+(a+b+c) \geq (k+1)(ab+bc+ca)$$

Because of $a+b+c=3$, we can mention $a+b+c=3 \geq ab+bc+ca$ which completes the problem.

**Generalization 37.2**
Let $a, b, c, d, k \in \mathbf{R}^+$ and $k \geq 1$ such that $a+b+c+d=4$. Then prove the following

$$a^k b + b^k c + c^k d + d^k a + 4k \geq (k+1)(ab+bc+cd+da)$$

*Proof.*
By homogenousity

$$a^k b+b^k c+c^k d+d^k a+4k \geq (k+1)(ab+bc+cd+da) = a^k b+b^k c+c^k d+d^k a+k(a+b+c+d)$$

$$= \sum_{cyc} \left( a^k b + \overbrace{b+b+\cdots+b}^{k-1} \right) + a+b+c+d$$

$$\overbrace{\geq}^{AM-GM} k(ab+bc+cd+da)+a+b+c+d \geq (k+1)(ab+bc+cd+da)$$

$$\rightarrow a+b+c+d \geq ab+bc+cd+da$$

Last expression can be shown like this

$$ab+bc+cd+da \overbrace{\geq}^{AM-GM} \frac{(a+b+c+d)^2}{4} = a+b+c+d$$

Problem has been finished.

**Application on Turkey TST 2017 Problem 5**
For all positive real numbers $a, b, c$ with $a + b + c = 3$, show that

$$a^3b + b^3c + c^3a + 9 \geq 4(ab + bc + ca)$$

*Proof.*
By placing $k = 3$ in Generalization 3.2, we get

$$a^3b + b^3c + c^3a + 9 \geq (k + 1)(ab + bc + ca) = 4(ab + bc + ca)$$

Which completes the proof.

**Generalization 38**
Let $a, b, c$ positive reals and $k \geq 1$ such that $a + b + c = \frac{k+1}{2k}$. Then prove the following

$$\frac{1}{kab + (k+1)c^2 + (k+1)c} + \frac{1}{kbc + (k+1)a^2 + (k+1)a} + \frac{1}{kca + (k+1)b^2 + (k+1)b} \geq \frac{1}{k(ab + bc + ca)}$$

*Proof.*

$$\sum_{cyc} \frac{k(ab + bc + ca)}{kab + (k+1)c^2 + (k+1)c} \geq 1$$

$$\frac{k(ab + bc + ca)}{kab + (k+1)c^2 + (k+1)c} \geq \frac{ab}{ab + bc + ca}$$

If the below inequality works, problem will be finished.

$$k(ab+bc+ca)^2 = ka^2b^2+kb^2c^2+kc^2a^2+2kabc(a+b+c) \geq ka^2b^2+(k+1)abc^2+(k+1)abc$$

$$kb^2c^2 + kc^2a^2 + 2kabc(a + b + c) \geq (k + 1)(abc^2 + abc)$$

Because of $a + b + c = \dfrac{k+1}{2k}$, we get

$$kb^2c^2+kc^2a^2+2kabc(a+b+c) \geq kb^2c^2+kc^2a^2+(k+1)(a+b+c) \geq (k+1)(abc^2+abc)$$

$$\to kb^2c^2 + kc^2a^2 \overbrace{\geq}^{AM-GM} 2kabc^2 \geq (k + 1)(abc^2)$$

$$2k \geq k + 1$$

This is true cause $k \geq 1$.

**Application on Turkey TST 2007 Problem 3**
Let $a, b, c$ be positive reals such that their sum is 1. Prove that

$$\frac{1}{ab + 2c^2 + 2c} + \frac{1}{bc + 2a^2 + 2a} + \frac{1}{ac + 2b^2 + 2b} \geq \frac{1}{ab + bc + ac}.$$

*Proof.*
By giving $k = 1$ in Generalization 4, we get the following

$$\frac{1}{ab + 2c^2 + 2c}+\frac{1}{bc + 2a^2 + 2a}+\frac{1}{ac + 2b^2 + 2b} \geq \frac{1}{k(ab + bc + ca)} = \frac{1}{ab + bc + ca}$$

Which completes the problem.

**Generalization 39**
Let $a, b, c, d, k, m, n, p$ be positive reals, n be positive integer with x positive divisors($d_1 < d_2 < \cdots < d_x$). Then prove that

$$\frac{\left(d_k a^2 + d_m b^2 + d_n c^2 + d_p d^2\right)\left(d_{x+1-p} a^2 + d_{x+1-n} b^2 + d_{x+1-k} c^2 + d_{x+1-m} d^2\right)}{(a+b)^2(c+d)^2} \geq n$$

*Proof.*
The arrangement of positive divisors is important because of

$$d_i . d_{x+1-i} = n$$

Rearrange inside of paranthesis by making these pairs.

$$\frac{\left(d_k a^2 + d_m b^2 + d_n c^2 + d_p d^2\right)\left(d_{x+1-k} c^2 + d_{x+1-m} d^2 + d_{x+1-n} b^2 + d_{x+1-p} a^2\right)}{(a+b)^2(c+d)^2}$$

$$\underbrace{\geq}_{Cauchy} \frac{\left(\sqrt{d_k d_{x+1-k}}ac + \sqrt{d_m d_{x+1-m}}bd + \sqrt{d_n d_{x+1-n}}cb + \sqrt{d_p d_{x+1-p}}da\right)^2}{(a+b)^2(c+d)^2}$$

$$= \frac{\left(\sqrt{n}(ac + bd + bc + ad)\right)^2}{(a+b)^2(c+d)^2} = \frac{n(a+b)^2(c+d)^2}{(a+b)^2(c+d)^2} = n$$

**Application on Turkey TST 2023 Problem 6**
Let $a, b, c, d$ be positive real numbers. What is the minimum value of

$$\frac{(a^2 + b^2 + 2c^2 + 3d^2)(2a^2 + 3b^2 + 6c^2 + 6d^2)}{(a+b)^2(c+d)^2}$$

*Proof.*
This problem is a special case of Generalization 5 in case of $n = 6$.

$$\frac{(a^2+b^2+2c^2+3d^2)(2a^2+3b^2+6c^2+6d^2)}{(a+b)^2(c+d)^2} \geq n = 6$$

**Generalization 40.1**
Let $a, b, c$ be reals such that $0 < x, y, z < p$. Then find the minimum value of

$$\frac{xyz(x+y+z)+(xy+yz+zx)(p^3-xyz)}{xyz\sqrt{p^3-xyz}}$$

*Proof.(Lokman Gökçe)*

$$\frac{xyz(x+y+z)+(xy+yz+zx)(p^3-xyz)}{xyz\sqrt{p^3-xyz}} = \frac{x+y+z}{\sqrt{p^3-xyz}}+(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})\sqrt{p^3-xyz}$$

$$\overbrace{\geq}^{AM-GM} 2\sqrt{(x+y+z)(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})} \geq 2\sqrt{9} = 6$$

We should prove that there exist positive reals for the expression's minimum value so that the equality case occurs. (Unless, the proof will be incompleted.)

This can be done by searching for $x$ values in case of $x = y = z$ and $\frac{x+y+z}{\sqrt{p^3-xyz}} = (\frac{1}{x}+\frac{1}{y}+\frac{1}{z})\sqrt{p^3-xyz}$.

$$\frac{3x}{\sqrt{p^3-x^3}} = \frac{3\sqrt{p^3-x^3}}{x} \Rightarrow x^3+x^2-p^3 = 0$$

If we define $P(x) = x^3 + x^2 - p^3$, then because of $P(0) = -p^3$, $P(p) = p^2$ we can mention by Intermediate Value Theorem for continuous functions, there exists a root of $P(x) = 0$ in between $(0, p)$ . Proof has been completed.

**Generalization 40.2**
Let $a_1, a_2, \cdots, a_n$ positive reels and $0 < a_1, a_2, \cdots, a_n < p$ . Then prove that

$$\frac{\prod a_1 . \left(\sum_{cyc} a_1\right) + \left(\sum a_1 a_2\right) . \left(p^n - \prod a_1\right)}{\prod a_1 \sqrt{p^n - \prod a_1}} \geq 2n$$

*Proof.*

**Application on Turkey TST 2020 Problem 8**
Let $x, y, z$ be real numbers such that $0 < x, y, z < 1$. Find the minimum value of

$$\frac{xyz(x+y+z) + (xy+yz+zx)(1-xyz)}{xyz\sqrt{1-xyz}}$$

*Proof.*
The problem is a special case of Generalization 38.2 where

$$n = 3, p = 1$$

with minimum 6.

# 12 Turkey EGMO Team Selection Test

**Generalization 41**
Let $a, b, c$ be positive reals such that $abc = 1$ , $a + b + c = p$ and

$$(\lambda ab + \theta a + \theta b - \theta p + \beta)(\lambda bc + \theta b + \theta c - \theta p + \beta)(\lambda ca + \theta c + \theta a - \theta p + \beta) \geq 0$$

expressions hold. Then prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{p\left(\beta+\sqrt{\beta^2+4\theta\lambda}\right)}{2\theta}+\frac{2\theta}{\beta+\sqrt{\beta^2+4\theta\lambda}}-\left(\frac{\beta+\sqrt{\beta^2+4\theta\lambda}}{2\theta}\right)^2$$

*Proof.*

**Application on Turkey EGMO TST 2019 Problem 2**
Let $a, b, c$ be positive reals such that $abc = 1$, $a + b + c = 5$ and

$$(ab+2a+2b-9)(bc+2b+2c-9)(ca+2c+2a-9) \geq 0$$

Find the minimum value of

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

*Proof.*
The problem is a special version of Generalization 39 where

$$\lambda = 1, \beta = 1, \theta = 2, p = 5$$

with minimum 5

**Generalization 42**
Let $x_1, x_2, \cdots, x_n, k$ be positive reals ($k \geq 1$) such that $\sum_{cyc} x_1 \geq \sum_{cyc} x_1^2$. Then prove that

$$\sum_{cyc} \frac{x_1^k+2k-1}{x_1^3+1} \geq kn$$

*Proof.*

$$x_1^k + 2k - 1 = x_1^k + \overbrace{1 + 1 + \cdots + 1}^{k-1} + k \overbrace{\geq}^{AM-GM} kx + k = k\left(x+1\right)$$

$$\sum_{cyc} \frac{x_1^k + 2k - 1}{x_1^3 + 1} \geq \sum_{cyc} \frac{k\left(x_1+1\right)}{\left(x_1+1\right)\left(x_1^2 - x_1 + 1\right)} = k\sum_{cyc} \frac{1}{x_1^2 - x_1 + 1}$$

$$\overbrace{\geq}^{Titu} k\frac{n^2}{\sum_{cyc} x_1^2 - \left(\sum_{cyc} x_1\right) + n} \geq kn$$

which finishes the problem.

**Application on Turkey EGMO TST 2014 Problem 4**
Let $x, y, z$ be positive reals such that $x^2 + y^2 + z^2 \leq x + y + z$. Then prove that

$$\frac{x^2+3}{x^3+1} + \frac{y^2+3}{y^3+1} + \frac{z^2+3}{z^3+1} \geq 6$$

*Proof.*
The problem is a special version of Generalization 40 where

$$n = 3, k = 2$$

# 13 Turkey Mathematical Olympiad 2nd Round

**Generalization 43**
Let $x_1, x_2, \cdots, x_{2k+1}$ be positive reals ($k \geq 1$). Then prove that

$$2\sqrt{\left(\sum_{cyc} x_1\right)\left(\sum_{cyc}\frac{1}{x_1}\right)} - \sqrt{\left(\sum_{i=1}^{n-1} x_i\right)\left(\sum_{j=2}^{n}\frac{1}{x_j}\right)} \geq 2\sqrt{2}+1+4(k-1)$$

*Proof.*

**Application on Turkey 2nd Round 2020 Problem 3**
If $x, y, z$ are positive real numbers find the minimum value of

$$2\sqrt{(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)} - \sqrt{\left(1+\frac{x}{y}\right)\left(1+\frac{y}{z}\right)}$$

*Proof.*
This problem is a special case of Generalization 41 where

$$k = 1$$

## 14 Japan Mathematical Olympiad Finals

**Generalization 44**
Find the maximum value of real number $k$ such that

$$\frac{a}{\lambda-1+9bc+k(b-c)^2}+\frac{b}{\lambda-1+9ca+k(c-a)^2}+\frac{c}{\lambda-1+9ab+k(a-b)^2} \geq \frac{1}{\lambda}$$

holds for all nonnegative reals $a, b, c$ satysfing $a+b+c=1$.

*Proof.*

**Application on Japan MO Finals 2014 Problem 5**
Find the maximum value of real number $k$ such that

$$\frac{a}{1+9bc+k(b-c)^2}+\frac{b}{1+9ca+k(c-a)^2}+\frac{c}{1+9ab+k(a-b)^2}\geq\frac{1}{2}$$

holds for all non-negative real numbers $a,\ b,\ c$ satisfying $a+b+c=1$.

*Proof.*
By Generalization 42 we know that the $k$ ($k=4$ given in the proof) holds for this problem too where

$$\lambda=2$$

which directly changes the minimum value depending on $\lambda$

**Generalization 45**
Let $(x_1)_1, (x_1)_2, \cdots, (x_1)_p, (x_2)_1, (x_2)_2, \cdots, (x_2)_p, \cdots\cdots, (x_n)_p$ be positive reals such that

$$M=\prod_{i=1}^{n}\left(\sum_{cyc}x_i^n+\lambda\right)$$

and

$$N=A\prod_{j=1}^{n}\left(\sum_{k=1}^{p}(x_p)_1\right)$$

hold. Then prove that

$$A\leq\frac{n^n\lambda}{[p(n-1)]^{p-1}}$$

*Proof.*

**Application on Japan MO Finals 2006 Problem 5**
For any positive real numbers $x_1,\ x_2,\ x_3,\ y_1,\ y_2,\ y_3,\ z_1,\ z_2,\ z_3$, find the maximum value of real number $A$ such that if

$$M = (x_1^3 + x_2^3 + x_3^3 + 1)(y_1^3 + y_2^3 + y_3^3 + 1)(z_1^3 + z_2^3 + z_3^3 + 1)$$

and

$$N = A(x_1 + y_1 + z_1)(x_2 + y_2 + z_2)(x_3 + y_3 + z_3)$$

then $M \geq N$ always holds.

*Proof.*
By Generalization 43 one can find maximum value of $A$ is $\dfrac{3}{4}$.

**Generalization 46.1**
Let $a_1, a_2, \cdots, a_n$ be positive reals ($n \geq 3$) such that $\sum_{cyc} a_i = \lambda$. Then prove that

$$\sum_{cyc-j} a_j \sqrt[n]{\lambda + a_{j+1} - a_{j-1}} \leq \frac{\lambda\,(\lambda + n - 1)}{n}$$

*Proof.*

**Generalization 46.2**
Let $a_1, a_2, \cdots, a_n, p$ be positive reals ($n \geq 3$) such that $\sum_{cyc} a_i = \lambda$. Then prove that

$$\sum_{cyc-j} a_j \sqrt[p]{\lambda + a_{j+1} - a_{j-1}} \geq \frac{\lambda\,(\lambda + p - 1)}{n}$$

*Proof.*

**Application on Japan MO Finals 2010 Problem 4**
Let $a, b, c$ be positive real numbers such that $a+b+c=1$. Prove the following inequality

$$a\sqrt[3]{1+b-c}+b\sqrt[3]{1+c-a}+c\sqrt[3]{1+a-b}\leq 1$$

*Proof.*
The problem is a special case of Generalization 44.2 where

$$n=3, p=3, \lambda=1$$

**Generalization 47**
Let $x_1, x_2, \cdots, x_n$ be positive reals $(n \geq 3)$. Then prove that

$$\sum_{i=1}^{n}\frac{x_i\left(\sum_{cyc}(x_1)-x_i\right)+\lambda}{\left(\sum_{cyc}(x_1)-x_i+\sqrt{\lambda}\right)^2}\geq 1$$

*Proof.*

**Application on Japan MO Finals 2010 Problem 4**
Let $x, y, z$ be positive reals. Then prove that

$$\frac{1+yz+zx}{(1+x+y)^2}+\frac{1+zx+xy}{(1+y+z)^2}+\frac{1+xy+yz}{(1+z+x)^2}\geq 1$$

*Proof.*
The problem is a special version of Generalization 45 where

$$n = 3, \lambda = 1$$

**Generalization 48**
Let $a, b, c, k$ be positive reals such that $a + b + c = \lambda$ ise

$$\frac{\lambda k + a}{\lambda - a} + \frac{\lambda k + b}{\lambda - b} + \frac{\lambda k + c}{\lambda - c} + 3k - 3 \le (k + 1)\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right)$$

*Proof.*

**Application on Japan MO Final 2004 Problem 4**
Let $a, b, c$ be positive reals such that $a + b + c = 1$. Then prove that

$$\frac{1 + a}{1 - a} + \frac{1 + b}{1 - b} + \frac{1 + c}{1 - c} \le 2\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right)$$

*Proof.*
Problem is a special case of Generalization 46 where

$$\lambda = 1, k = 1$$

# 15 Iran Mathematical Olympiad 3rd Round

**Generalization 49**
Let $a_1, a_2, \cdots, a_n$ be positive reals ($n \ge 3$) such that $\sum_{cyc} a_1 a_2 = \lambda$. Then prove that

$$\prod_{cyc-j}\left(\sqrt{\frac{\prod a_1}{a_j}}+\frac{\lambda}{2.\sum_{k=j}^{j+n-3}\left(\frac{a_k}{\sqrt{\frac{\prod a_1}{a_{k-1}a_ka_{k+1}}}}\right)+a_{j-2}a_{j-1}\sqrt{\frac{a_j}{\prod a_1}}}\right)\geq 2^n\sqrt{\left(\prod a_1\right)^{n-1}}$$

*Proof.*

**Application on Iran 3rd Round 2018 Problem A.1**
For positive real numbers $a, b, c$ such that $ab+ac+bc=1$ prove that

$$\prod_{cyc}(\sqrt{bc}+\frac{1}{2a+\sqrt{bc}})\geq 8abc$$

*Proof.*
The problem is a special case of Generalization 47 where

$$n=3,\lambda=1$$

**Generalization 50**
Let $a_1, a_2, a_3, a_4$ be positive reals such that $a+b+c+d=p$. Then prove that

$$\sum_{cyc}\frac{\lambda a_ia_{i+1}}{a_i^2-\frac{k-\beta}{k}a_i+\frac{k+\lambda}{k}}\leq\frac{\lambda p^2k}{(k-\beta)+4\lambda}$$

*Proof.*

**Application on Iran 3rd Round 2021 Problem A.1**
Positive real numbers $a, b, c$ and $d$ are given such that $a + b + c + d = 4$ prove that

$$\frac{ab}{a^2 - \frac{4}{3}a + \frac{4}{3}} + \frac{bc}{b^2 - \frac{4}{3}b + \frac{4}{3}} + \frac{cd}{c^2 - \frac{4}{3}c + \frac{4}{3}} + \frac{da}{d^2 - \frac{4}{3}d + \frac{4}{3}} \leq 4$$

*Proof.*
Problem is a special version of Generalization 48 where

$$k = 3, \lambda = 1, \beta = 1, p = 4$$

## 16 China Team Selection Test

**Generalization 51.1**
Let $x, y, z$ be positive reals such that $x + y + z = \lambda xyz$. Then prove that

$$\sum_{cyc} \left(x^{3p+1} \left(\lambda yz - 1\right)\right) \geq \frac{2\left(3\sqrt{3}\right)^{p+1}}{\sqrt{\lambda^{3p+1}}}$$

*Proof.*

**Generalization 51.2**
Let $x_1, x_2, \cdots, x_n, \lambda, p, n$ be positive reals $(n \geq 3)$ such that $\sum_{cyc} x_1 = \lambda \prod x_1$. Then prove that

$$\sum_{cyc}\left(x^{pn+1}\left(\lambda\frac{\prod x_i}{x_1}-1\right)\right)\geq\frac{(n-1)\sqrt[n-1]{n^{n(p+1)}}}{\sqrt[n-1]{\lambda^{pn+1}}}$$

*Proof.*

**Application on China TST 2003 Problem 3.1**
Let $x, y, z$ be positive reals such that $x+y+z=xyz$. Find the minimum value of the following expression

$$x^7(yz-1)+y^7(zx-1)+z^7(xy-1)$$

*Proof.*
The problem is a special case of Generalization 49.2 where

$$n=3, p=2, \lambda=1$$

# 17 Romania Team Selection Test

**Generalization 52**
$(n, k\in\mathbf{R}^+)$ $x_1, x_2, x_3, x_4$ positive reals with sum $\lambda$. Then prove that

$$\sum_{1\leq i<j\leq 4}(x_i+x_j)^p\sqrt[k]{x_ix_j}\leq\frac{3^p.\lambda^{2p}}{2^{2p}.\sqrt[2k]{\left(\sum_{1\leq i<j\leq 4}x_ix_j\right)^{kp-2}}}$$

*Proof.*

**Application on Romania TST 2019 Problem 5.1**
Determine the largest value the expression

$$\sum_{1\le i<j\le 4} (x_i + x_j)\sqrt{x_i x_j}$$

may achieve, as $x_1, x_2, x_3, x_4$ run through the non-negative real numbers, and add up to 1. Find also the specific values of this numbers that make the above sum achieve the asked maximum.

*Proof.*
The problem is a speacial version of Generalization 50 where

$$n = 4, p = 1, k = 2$$

with equality case if and only if

$$x_1 = x_2 = \cdots = x_n = \frac{\lambda}{n}$$

# 18 USA Team Selection Test

Let us see that the generalizations of IMO 1995 2 is same with USA TST 2010 2.

**Generalization 53.1**
Let $a_1, a_2, \cdots, a_n, n, p$ be positive reals ($n \ge 2$ ,$p \ge 1$) such that $\prod a_1 = 1$. Then prove that

$$\sum_{cyc}\left(\frac{1}{a_1^{2p+1}\left(x_1.\dfrac{\prod a_1}{a_1 a_n} + x_2.\dfrac{\prod a_1}{a_1 a_2}\right)^p}\right) \ge \frac{n}{(x_1 + x_2)^p}$$

*Proof.*

**Generalization 53.2**
Let $a_1, a_2, \cdots, a_n, n, p, k$ be positive reals ($n \geq 2$ ,$p \geq 1$) such that $\prod a_1 = k$. Then prove that

$$\sum_{cyc} \left( \frac{k^{p+1}}{a_1^{2p+1} \left( x_1 . \frac{\prod a_1}{a_1 a_n} + x_2 . \frac{\prod a_1}{a_1 a_2} \right)^p} \right) \geq \frac{n \sqrt[n]{k^{n-1}}}{(x_1 + x_2)^p}$$

*Proof.*

**Application on USA TST 2010 Problem 2**
Let $a, b, c$ be positive reals such that $abc = 1$. Then prove that

$$\frac{1}{a^5(b+c)^2} + \frac{1}{b^5(c+a)^2} + \frac{1}{c^5(a+b)^2} \geq \frac{1}{3}$$

*Proof..*
The problem is a special case of Generalization 3 where

$$n = 3, p = 2, k = 1, x_1 = 1, x_2 = 2$$

# 19 USA Mathematical Olympiad

**Generalization 54**
Let $a_1, a_2, \cdots, a_n$ be positive reals. Then prove that

$$\sum_{cyc} \frac{\left((k+1)a_1 + ka_2 + ka_3\right)^2}{(k+1)a_1^2 + k(a_2 + a_3)^2} \leq 6k + \frac{n}{k+1} + 1$$

*Proof.*
Inequality is homogenous. WLOG assume that $\sum_{cyc} a_1 = 1$.

$$\sum_{cyc} \frac{((k+1)a_1 + k(a_2+a_3+\cdots+a_n))^2}{(k+1)a_1^2 + k(a_2+a_3+\cdots+a_n)^2} = \sum_{cyc} \frac{(a_1+k)^2}{(k+1)a_1^2 + k(1-a_1)^2} = \sum_{cyc} \frac{(a_1+k)^2}{(2k+1)a_1^2 - 2a_1k + k}$$

$$= \sum_{cyc} \left( \frac{1}{2k+1} + \frac{2a_1k + \dfrac{2a_1k}{2k+1} + k^2 - \dfrac{k}{2k+1}}{(2k+1)a_1^2 - 2a_1k + k} \right) = S$$

By making some changes on denominator.

$$(2k+1)a_1^2 - 2a_1k + k = (2k+1)a_1^2 - 2a_1k + \frac{k^2}{2k+1} + k - \frac{k^2}{2k+1} \overbrace{\geq}^{AM-GM} 2a_1k - 2a_1k + k - \frac{k^2}{2k+1} = k - \frac{k^2}{2k+1}$$

$$S \leq \sum_{cyc} \left( \frac{1}{2k+1} + \frac{2a_1k + \dfrac{2a_1k}{2k+1} + k^2 - \dfrac{k}{2k+1}}{k - \dfrac{k^2}{2k+1}} \right)$$

$$= \frac{n}{2k+1} + \frac{2k(\sum_{cyc} a_1) + \dfrac{2k}{2k+1}(\sum_{cyc} a_1) + nk^2 - \dfrac{nk}{2k+1}}{k - \dfrac{k^2}{2k+1}}$$

$$\frac{n}{2k+1} + \frac{2k + \dfrac{2k}{2k+1} + nk^2 - \dfrac{nk}{2k+1}}{k - \dfrac{k^2}{2k+1}} = \frac{n}{2k+1} + \frac{\dfrac{k(2-n)}{2k+1} + k(2+nk)}{k - \dfrac{k^2}{2k+1}}$$

$$= \frac{n}{2k+1} + \frac{\frac{2-n}{2k+1} + 2 + nk}{\frac{k+1}{2k+1}} = \frac{n}{2k+1} + \frac{2nk^2 + nk - n + 4k + 4}{k+1}$$

$$= \frac{n}{2k+1} + \frac{n(2k^2 + k - 1)}{k+1} + 4 = \frac{n}{2k+1} + \frac{n(2k-1)(k+1)}{k+1} + 4$$

$$= \frac{n}{2k+1} + n(2k-1) + \frac{n}{2k+1} + 4$$

Problem has been solved.

**Application on USAMO 2003 Problem 5**
Let $a, b, c$ be positive reals. Then prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \leq 8$$

*Proof.*
By substituting $k = 1$ in Generalization 6, we get

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \leq 6k + \frac{n}{k+1} + 1 = 8$$

**Generalization 55**
Let $a_1, a_2, \cdots, a_{p+1}$ be positive reals and $p$ be a nonnegative integer. Then prove that

$$\prod \left(a^{2p+1} - a^p + p + 1\right) \geq \left(\sum_{cyc} a_1\right)^{p+1}$$

*Proof.*

**Application on USAMO 2004 Problem 5**
Let $a, b, c > 0$. Prove that

$$\left(a^5 - a^2 + 3\right)\left(b^5 - b^2 + 3\right)\left(c^5 - c^2 + 3\right) \geq (a + b + c)^3$$

*Proof.*
The problem is a special case of Generalization 53 where

$$p = 2$$

## 20 Korea Mathematical Olympiad Finals

**Generalization 56**
Let $x, y, z$ be positive reals such that $x + y + z = 1$. Then prove that

$$\frac{2\left(1 + xy + yz + zx\right)\left(\theta\left(x^2 + y^2 + z^2\right) + \dfrac{\lambda}{9}\right)}{\sqrt{\lambda\theta\left(\lambda + \theta\right)}\left(x + y\right)\left(y + z\right)\left(z + x\right)} \geq \left(\sum_{cyc} \frac{x\sqrt{x + 1}}{\sqrt[4]{\theta + \lambda x^2}}\right)^2$$

*Proof.*

**Application on Korea MO Final 2014 Problem 1**
Let $x, y, z$ be positive reals such that $x + y + z = 1$. Then prove that

$$\frac{\left(1 + xy + yz + zx\right)\left(1 + 3x^2 + 3y^2 + 3z^2\right)}{9\left(x + y\right)\left(y + z\right)\left(z + x\right)} \geq \left(\sum_{cyc} \frac{x\sqrt{x + 1}}{\sqrt[4]{3 + 9x^2}}\right)^2$$

*Proof.*
The problem is a special version of Generalization 54 where

$$\lambda = 9, \theta = 3$$

## 21 Belarus Team Selection Test

**Generalization 57**
For $x, y, z \in \mathbf{R}^+$ such that $x + y + z = 2k$ prove that

$$\sum_{cyc} \frac{(x-k)^2}{y} \geq \frac{1}{4}\left(\frac{x^2+y^2}{x+y} + \frac{y^2+z^2}{y+z} + \frac{z^2+x^2}{z+x}\right)$$

*Proof.*
Let us multiply both sides by 2.

$$\left(\frac{(x-k)^2}{y} + \frac{(y-k)^2}{z}\right) + \left(\frac{(y-k)^2}{z} + \frac{(z-k)^2}{x}\right) + \left(\frac{(z-k)^2}{x} + \frac{(x-k)^2}{y}\right)$$

$$\overbrace{\geq}^{Bergstorm} \sum_{cyc} \frac{(x+y-2k)^2}{y+z} = \sum_{cyc} \frac{z^2}{y+z} \geq \frac{1}{2}\left(\sum_{cyc} \frac{x^2+y^2}{x+y}\right)$$

$$\rightarrow \sum_{cyc} \frac{2z^2 - y^2 - z^2}{y+z} = \sum_{cyc} \frac{z^2-y^2}{y+z} = \sum_{cyc} z - y = 0 \geq 0$$

**Application on Belarus TST 2017 Problem 4.2**
Given that $x$, $y$, $z$ are positive real numbers satisfying $x + y + z = 2$, prove the

inequality

$$\frac{(x-1)^2}{y}+\frac{(y-1)^2}{z}+\frac{(z-1)^2}{x}\frac{1}{4}\left(\frac{x^2+y^2}{x+y}+\frac{y^2+z^2}{y+z}+\frac{z^2+x^2}{z+x}\right)$$

*Proof.*
Replacing $k=1$ in Generalization 8, we get

$$\frac{(x-1)^2}{y}+\frac{(y-1)^2}{z}+\frac{(z-1)^2}{x}\geq\frac{1}{4}\left(\frac{x^2+y^2}{x+y}+\frac{y^2+z^2}{y+z}+\frac{z^2+x^2}{z+x}\right)$$

Problem has been completed.

## 22 Macedonia National Olympiad

**Generalization 58**
Let $a_1, a_2, \cdots, a_n$ be reels ($k \geq 1, n = 3, 4$) and $\sum_{cyc} a_1 = x$. Then prove that

$$\sum_{cyc}\frac{a_j^k+k-1}{a_{j+1}+k-1}\geq\frac{nxk}{x+nk-n}$$

*Proof.*

$n = 3$ version was given in **[5]** with it's proof. Let us show that $n = 4$ also works.

$$\sum_{cyc}\frac{a^k+k-1}{b+k-1}=\sum_{cyc}\frac{a^k+\overbrace{1+1+\cdots+1}^{k-1}}{b+k-1}\overbrace{\geq}^{AM-GM}\sum_{cyc}\frac{ka}{b+k-1}$$

$$=k\left(\frac{a}{b+k-1}+\frac{b}{c+k-1}+\frac{c}{d+k-1}+\frac{d}{a+k-1}\right)$$

$$\overbrace{\geq}^{Titu} k\frac{(a+b+c+d)^2}{ab+bc+cd+da+(k-1)(a+b+c+d)} \geq k\frac{(a+b+c+d)^2}{\frac{(a+b+c+d)^2}{4}+(k-1)(a+b+c+d)}$$

$$= k\frac{a+b+c+d}{\frac{a+b+c+d}{4}+k-1} = k\frac{4(a+b+c)}{a+b+c+4k-4}$$
$$= \frac{4xk}{x+4k-4}$$

Which completes the problem.

**Application on Macedonia MO Problem 1.2**
Let $a, b, c$ be positive reals such that $a+b+c=3$. Then prove that

$$\frac{a^3+2}{b+2}+\frac{b^3+2}{c+2}+\frac{c^3+2}{a+2} \geq 3$$

*Proof.*
The problem is a special version of Generalization 56 where

$$n=3, k=3$$

# 23 References


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**[3]** Mathematical Association of America- AMC (maa.org)
**[4]** Inequalities, G.H. Hardy, J.E. Littlewood, George Pólya
**[5]** Some Algebraic Inequalities, Hüseyin Emekçi
**[6]** Art of Problem Solving Forum