It should be noted that the short cycle generators can also be calculated directly (but less simply) from the function f. For example, for $n = 4$, the 1-cycle must satisfy

$$x = \mathrm{f}(x) = \tfrac{1}{4}(x+1) + 2^3,$$

so that $x = 11$.

The general case

The general case, for packs of any size, is much more difficult to solve, and I have made little progress. One result I have proved, however, is that for a pack of $6k + 1$ cards there is always a 1-cycle consisting of the number $4k + 1$. For example, if there are 67 cards, (45) is a 1-cycle.

Note added in proof. Since this article went to press, I have found that, for packs of size 2^n-1, the number of shuffles required is the l.c.m. of the numbers less than or equal to n. The proof is similar to that given in the article for packs of size 2^n. In would be pleased to give further details to any reader who is interested.

L. W. GATES

Gillingham Technical High School, Pump Lane, Gillingham, Kent ME8 7AJ

Last words on adventitious angles

The story so far

The June 1975 *Gazette* (**59**, No. 408, 98–106) carried an article by Colin Tripp about the so-called "adventitious angles" problem†: in the isosceles triangle in Fig. 1, for certain given values of the angles marked a, b, c, to find θ. Correspondence to which this gave rise was summarised in a "progress report" in March 1977 (**61**, No. 415, 55–58). This cleared up certain points raised by Tripp, using trigonometric methods; and its publication also had the desired effect of stimulating some further response, which has settled most of the outstanding questions. It seems now to be an appropriate time for the editor to produce a "final summary" for those readers who have taken an interest in the problem.

† Students of *Gazette* history will be interested in a remark by Mr Parry that the particular case (20, 60, 50; 30) was set as a problem by E. M. Langley in *Gazette* **11**, 173 (Note 644, No. 160, October 1922); this became known as "Langley's problem". Solutions by twelve readers were published in **11**, 321–323 (No. 164, May 1923). The pace of life was not always slower 50 years ago!

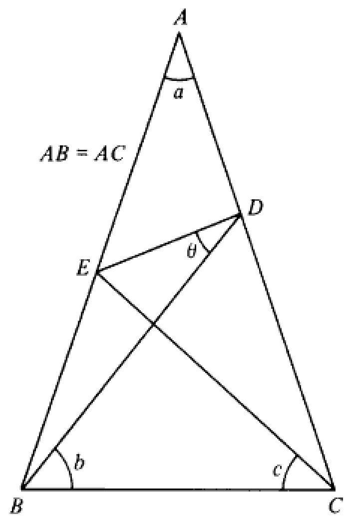


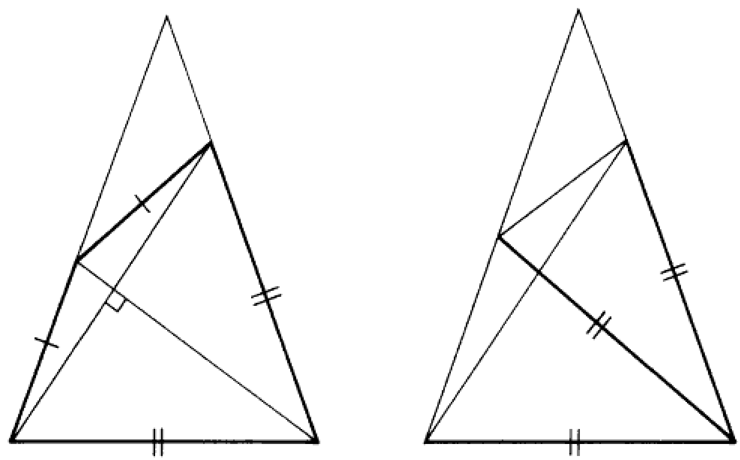
FIGURE 1.

Tripp's article was concerned mainly with the 'classical' form of the problem in which a, b, c and θ are all integral numbers of degrees, although he also suggested two generalisations which were picked up by some correspondents. We will refer to a solution by the symbol $(a, b, c; \theta)$. (In earlier articles the notation (a, b, c) has been used, with θ described as the "derived angle"; but it is convenient to include the value of θ in the description of the set.) Tripp identified two particular families of solutions, which he called "kites" and "fans" (Fig. 2), given by $(a, 45 + \frac{1}{4}a, 45 - \frac{1}{4}a; 45 - \frac{3}{4}a)$ and $(a, 45 + \frac{1}{4}a, a; \frac{1}{2}a)$ respectively, and showed how various other solutions could be related to these. This left unresolved 20 suspected solutions, subsequently verified trigonometrically, which were listed in the second article as follows:

$X_1(12,57,33;15)$	$X_1'(12,57,42;24)$	$Y_1(12,42,18;12)$	$Y_1'(12,42,30;24)$
$X_2(12,69,21;\ 3)$	$X_2'(12,69,66;48)$	$Y_2(12,66,42;12)$	$Y_2'(12,66,54;24)$
$X_3(72,48,42;24)$	$X_3'(72,48,24;\ 6)$	$Y_3(12,72,42;\ 6)$	$Y_3'(12,72,66;30)$
$X_4(72,51,39;\ 9)$	$X_4'(72,51,42;12)$	$Y_4(72,39,21;12)$	$Y_4'(72,39,27;18)$
		$Y_5(72,42,24;12)$	$Y_5'(72,42,30;18)$
		$Y_6(120,24,12;\ 6)$	$Y_6'(120,24,18;12)$

"Hovering between two worlds . . ."

"Empty contents into a large saucepan or preserving pan and add 1¼ pints of water. Mix in two 1 kilo bags of sugar to make 7 lbs of delicious Marmalade; should a more tangy product be required add only 4 lbs of sugar which will produce 6 lbs of Marmalade." From the instructions on the label of Beach's seville oranges (per Christian Puritz).



"Kite" "Fan"

FIGURE 2.

The separation into X and Y classes was suggested by trigonometric considerations, but it transpires that it is not important geometrically. The pairs denoted by the same letter and suffix, with and without primes, are pairs of "cyclic complements"; if either of the pair is established, the other can be deduced from it.

The further contributions, which fall into three categories, will now be described.

Geometrical relationships

The two correspondents who tackled the geometrical problem directly were Mr C. F. Parry of Burghfield Common, Berkshire and Mr J. C. E. Wren of Holmes Chapel, Cheshire.

In his original article Tripp identified three possible ways in which solutions could be obtained from each other:

(1) using properties of isosceles (and equilateral) triangles;

(2) using a cyclic quadrilateral (the symmetric "cyclic complement" relationship);

(3) using two cyclic quadrilaterals.

These are illustrated by the diagrams numbered (1)–(3) in Fig. 3. Parry gave the general forms of these and added two of his own, numbered (4) and (5), in each of which the centre of the escribed circle of the shaded triangle lies on one side of the isosceles triangle. Wren also pointed out (a fact suggested in passing by Tripp) that (3) can be reversed, giving (on interchanging primed and unprimed letters) a further deduction $(\bar{3})$. In the following table, the original solution $(a, b, c; \theta)$ gives rise to another solution $(a, b', c'; \theta')$. Using these, the chains shown in Fig. 4 can be established.

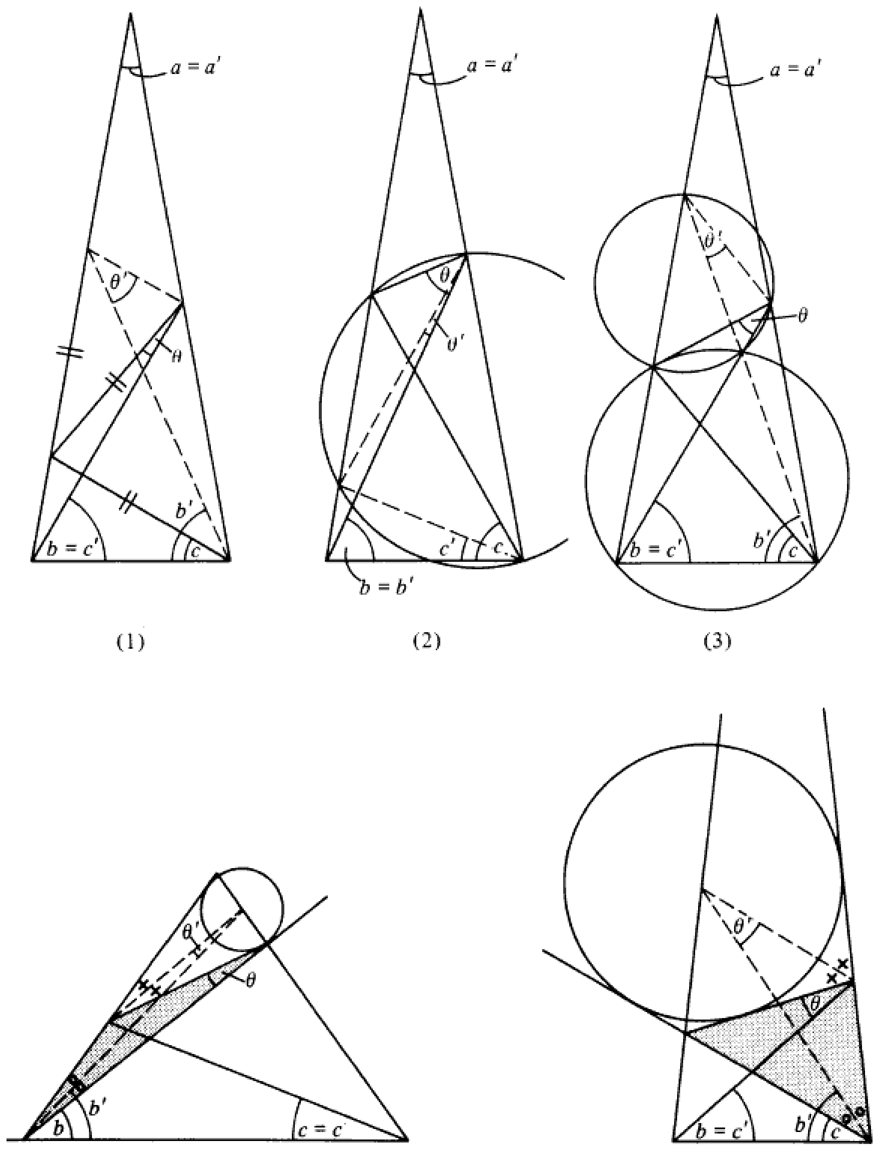


FIGURE 3.

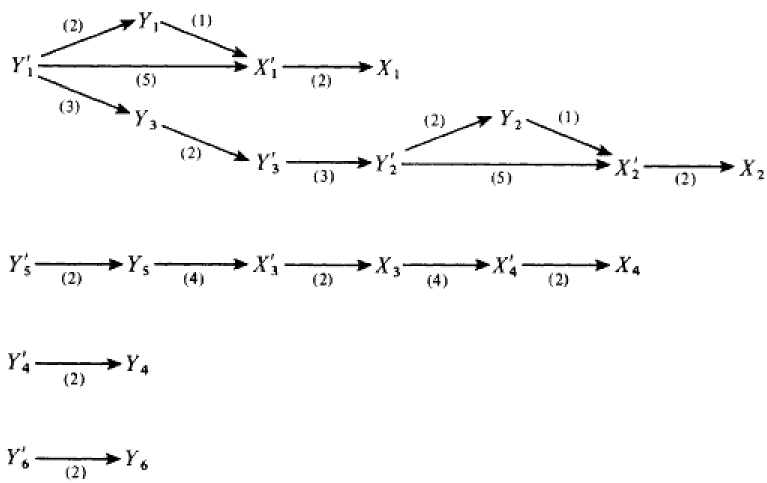


FIGURE 4.

Condition	b'	c'	θ'
(1) $b = a + c + \theta$	$45 + \frac{1}{4}a + \frac{1}{2}c$	b	$\frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}\theta$
(2) None	b	$b - \theta$	$b - c$
(3) $a + b = c + \theta$	$90 + \frac{1}{2}a - \theta$	b	$c - \theta$
($\bar{3}$) $\theta + 180 = 2b + c$	c	$b + \frac{1}{2}a + c - 90$	$90 - b + \frac{1}{2}a$
(4) $2b = a + \theta$	$45 + \frac{1}{4}\theta$	c	$\frac{1}{2}\theta$
(5) $a + b = c + \theta$	$45 - \frac{1}{4}a + \frac{1}{2}c$	b	$\frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}\theta$

The problem is therefore reduced to finding geometric proofs for Y_1', Y_4', Y_5', Y_6' (or their equivalent cyclic complements). Parry gave independent proofs of Y_1', Y_4', Y_5, Y_6, of which I reproduce here his proof of Y_1'; the others use properties of the regular pentagon and of similar figures, but for reasons of space they cannot be given here.

PROOF OF Y_1'. In Fig. 5, draw XY and XZ perpendicular to BC and BE; bisect XC and XD at P and Q, and let QZ meet EX at R. The circle XYC has centre P, and it follows that

$$XQ = XP = XY = XZ.$$

Hence XQZ is isosceles and, since $\angle BXZ = 48°$, $\angle XQZ = \angle XZQ = 24°$.

But $\angle EXZ = 90° - 66° = 24°$. Therefore RXZ is isosceles, and hence R is the centre of the circle EXZ. Therefore R bisects XE; and since Q bisects XD, QR is parallel to DE and $\theta = \angle EDX = \angle RQX = 24°$, thus demonstrating the adventitious set Y_1' (12, 42, 30; 24).

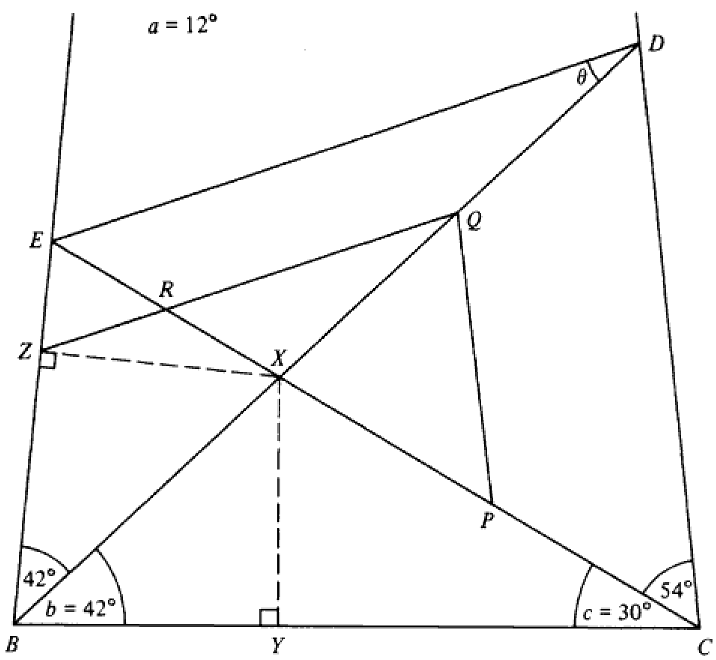


FIGURE 5.

Trigonometric generalisations

Mr Michael Behrend of Cambridge wrote describing some adaptations and extensions of the trigonometric relation given in Tripp's article, which he put into the form

$$\sin b \sin c \sin(b-\theta) - \cos^2\tfrac{1}{2}a \sin(c-\theta) = 0.$$

Writing $x=-b$, $y=c$, $z=b-\theta$, this takes the symmetric form

$$\sin x \sin y \sin z + \cos^2\tfrac{1}{2}a \sin(x+y+z) = 0.$$

Behrend pointed out that taking the cyclic complement is equivalent to exchanging y and z. Replacing products by sums gives

$$\sin(-x+y+z) + \sin(x-y+z) + \sin(x+y-z) + \sin(x+y+z) \\ + \sin(x+y+z-a) + \sin(x+y+z+a) = 0. \qquad (1)$$

He continues: "For adventitious angles as defined by Tripp x, y, z and θ are all multiples of 1°, and he later suggests that other submultiples of 2π might be investigated. This leads on to a more general problem: given an integer $m \geqslant 3$, it is required to find non-trivial relations of the form

$$\sum_{r\in R} a_r \sin(2r\pi/m) = 0,$$

where R is some finite set of integers, and the a_r are integers. Clearly we may suppose that the $\sin(2r\pi/m)$ are all positive, hence that r ranges from 1 to $\frac{1}{2}(m-1)$ if m is odd, and from 1 to $[\frac{1}{4}m]$ if m is even. Further, if m_1 is odd then $\sin(2r\pi/m)$ takes the same set of values whether $m = m_1$ or $m = 2m_1$. So when convenient we may assume that m is even. The main result ... is

THEOREM 1. *Let m be an integer $\geqslant 3$ and let $\phi(m)$ be Euler's function. Then*

(i) *the* $\sin(2r\pi/m)$ *with* $r = 1, 2, \ldots, \frac{1}{2}\phi(m)$ *are linearly independent over the integers;*

(ii) *any* $\sin(2r\pi/m)$ *with r an integer is an integral linear combination of those in* (i).

EXAMPLE 1. Take $m = 30$. Then $\sin(2r\pi/m)$ takes 7 distinct positive values. Since $\frac{1}{2}\phi(30) = 4$, the theorem says that $\sin 12^\circ$, $\sin 24^\circ$, $\sin 36^\circ$ and $\sin 48^\circ$ are linearly independent, while the other 3 values are integral linear combinations of these. In fact we find

$$\sin 60^\circ = \sin 12^\circ + 2\sin 24^\circ + \sin 36^\circ - \sin 48^\circ,$$
$$\sin 72^\circ = \sin 12^\circ + \sin 48^\circ,$$
$$\sin 84^\circ = \sin 24^\circ + \sin 36^\circ.$$

Note that, because the terms on the right are linearly independent, any integral linear relation between the 7 values results from an integral linear combination of these 3 equations. The same principle applies to any value of m.

EXAMPLE 2. Take $m = 4p$ where p is an odd prime. It can be verified that

$$1 = \sin\frac{2p\pi}{4p} = 2\sum_{j=1}^{(p-1)/2}(-1)^{j-1}\sin\frac{2(p-2j)\pi}{4p}.$$

Here there are p distinct positive values, and $\frac{1}{2}\phi(m) = p - 1$, so this relation is essentially the only one possible.

EXAMPLE 3. If m ($\geqslant 3$) is (i) a prime or (ii) twice a prime or (iii) a power of 2, then $\sin(2r\pi/m)$ takes exactly $\frac{1}{2}\phi(m)$ distinct positive values; hence no integral linear relations exist. Conversely, such relations exist for all other values of m.

THEOREM 2. *Suppose m is even. For each factorisation $m = 2c(2d+1)$ with $c \geqslant 2$ and $d \geqslant 1$, there exist relations*

$$\sin\frac{2k\pi}{m} + \sum_{j=1}^{d}(-1)^{j-1}\left(\sin\frac{2(jc-k)\pi}{m} - \sin\frac{2(jc+k)\pi}{m}\right) = 0,$$

for $k = 1, 2, \ldots, [\frac{1}{2}c]$.

PROOF. The stated values of k are all those that give arguments in the range 0 to $\frac{1}{2}\pi$. No further information is gained by taking k beyond these limits. To prove the identity, write the left side as

$$\sin\frac{2k\pi}{m}\left\{1+2\sum_{j=1}^{d}\cos\left(j\pi+\frac{j\pi}{2d+1}\right)\right\}$$

$$=\sin\frac{2k\pi}{m}\sum_{j=-d}^{d}\cos\left(j\pi.\frac{2d+2}{2d+1}\right).$$

Summing by the standard formula for a trigonometric arithmetic progression we find a factor $\sin(d+1)\pi=0$.

The pattern is shown by the following examples, in which (r) stands for $\sin(2r\pi/m)$.

EXAMPLE 4. $m=60$, $c=6$, $d=2$.

$$(1)+(5)-(7)-(11)+(13)=0,$$
$$(2)+(4)-(8)-(10)+(14)=0,$$
$$(3)+(3)-(9)-\ \ (9)+(15)=0.$$

EXAMPLE 5. $m=70$, $c=7$, $d=2$.

$$(1)+(6)-\ \ (8)-(13)+(15)=0,$$
$$(2)+(5)-\ \ (9)-(12)+(16)=0,$$
$$(3)+(4)-(10)-(11)+(17)=0.$$

The columns read alternately down and up, with multiples of c omitted. Numbers in the bottom row are repeated if c is even, but not if c is odd.

The problem arises: assuming as we may that m is even, does Theorem 2 suffice to give all the linear relations stated in Theorem 1? I believe that the answer is "yes", but I have not found a proof applicable to all values of m. Can the reader supply one? At any rate, it is easy to check that Theorem 2 suffices when $m=360$ (cases with $d=1,2$ and 4 being required), and the proof via equation (1) of all the identities found by Tripp reduces to routine calculation."

Tripp's generalisations

Dr John F. Rigby of University College, Cardiff (who also contributed to the earlier discussion) and Dr Paul Monsky of Brandeis University, Mass. gave their attention to the two generalisations suggested in §9 of Tripp's original article. One of these was to drop the condition that the triangle ABC be isosceles; this concentrates attention on the quadrangle $BCDE$

shown in Fig. 6. The problem is then to find all such quadrangles for which the angles B, b, C, c, θ are integral numbers of degrees (or, more generally, rational multiples of 180°).

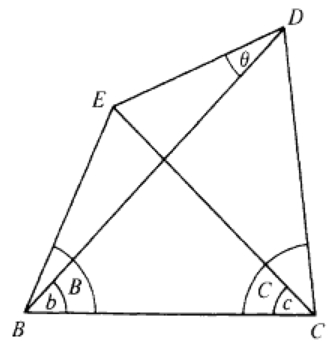


FIGURE 6.

Monsky transforms the problem into the form shown in Fig. 7. It is easy to see that the angles θ_i satisfy $\sum_{i=1}^{i=6} \theta_i = 180°$ and that

$$\sin\theta_1 \sin\theta_2 \sin\theta_3 = \sin\theta_4 \sin\theta_5 \sin\theta_6.$$

His classification of the solutions of these, obtained using arguments from trigonometry, algebra and number theory, leads at once to solutions of the problem in Tripp's original form, since

$$B = \theta_1 + \theta_4, \quad b = \theta_1, \quad C = 180° - (\theta_1 + \theta_6), \quad c = \theta_3 + \theta_5, \theta = \theta_3.$$

By considering the particular cases in which $B = C$, i.e.

$$(\theta_1 + \theta_4) + (\theta_1 + \theta_6) = 180°,$$

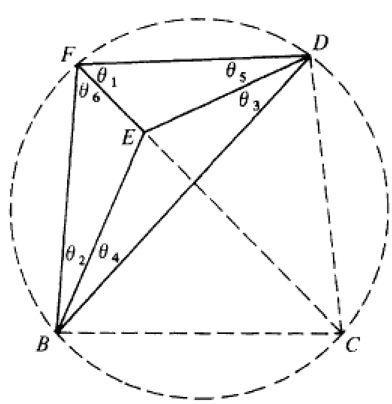


FIGURE 7.

Monsky deals with Tripp's other generalisation to "N-adventitious angles"—solutions of the isosceles triangle problem in which the angles are integral multiples of $180/N$ degrees for values of N other than factors of 180. He shows that, apart from kites and fans (which clearly give infinitely many solutions), there are only 8 other solutions. These, together with the $53-(2\times 13+1)=26$ of Tripp's adventitious sets which are not kites or fans, give in all a total of 34 N-adventitious sets—17 pairs of cyclic complements—which we list here; the columns containing Monsky's solutions are indicated with an asterisk.

			*								*	*					*
N	18	18	24	30	30	30	30	30	30	36	42	42	60	60	60	60	84
a	2	2	6	2	2	2	12	12	20	4	2	12	4	4	24	24	24
b	6	7	6	7	11	12	7	8	4	13	13	8	19	23	13	17	23
c	3	5	5	3	7	7	4	4	2	5	11	5	11	7	7	13	19
θ	1	1	4	2	2	1	2	1	1	1	8	4	5	1	4	3	13
a'	2	2	6	2	2	2	12	12	20	4	2	12	4	4	24	24	24
b'	6	7	6	7	11	12	7	8	4	13	13	8	19	23	13	17	23
c'	5	6	2	5	9	11	5	7	3	12	5	4	14	22	9	14	10
θ'	3	2	1	4	4	5	3	4	2	8	2	3	8	16	6	4	4

Rigby, in his discussion of the quadrangle problem, brings together a number of unexpected geometrical relationships. It seems appropriate to give his account in full in a separate article, which follows this summary. In his §9 he makes further reference to Monsky's results.

D.A.Q.

Adventitious quadrangles: a geometrical approach

J. F. RIGBY

1. Introduction

A quadrangle has four vertices, of which no three are collinear, and six sides joining the vertices in pairs. If the angle between each pair of the six sides is an integral multiple of π/n radians, n being an integer, the quadrangle is said to be *n-adventitious* [1]. A quadrangle is *adventitious* if it is n-adventitious for some n. For example, the quadrangle $BCDE$ in Fig. 1 (the original adventitious quadrangle from which all the discussion started in [1]) is 18-adventitious. Various problems are posed in [1]; in a suitably generalised form these problems can be summarised as: find all adventitious quadrangles *and prove their existence by elementary geometry.*