The adventitious angles problem: a progress report

In the *Gazette* for June 1975 (**59**, No. 408, 98–106) Colin Tripp showed how the familiar problem illustrated in Fig. 1 (in the isosceles triangle, for certain given values of the angles marked a, b, c, to find θ) could be tackled for most of the possible triples (a,b,c) revealed by a computer investigation. At the end of his study, 20 cases remained open; computation suggested that they were adventitious, but no proof had been found. These are listed in the table as 10 pairs of "cyclic complements", which Tripp proved to be equivalent; if one of the pair is adventitious, so is its cyclic complement.

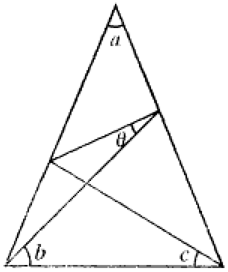


FIGURE 1.

	a	b	c	θ		a	b	c	θ
X_1	12	57	33	15	X_1'	12	57	42	24
X_2	12	69	21	3	X_2'	12	69	66	48
X_3	72	48	42	24	X_3'	72	48	24	6
X_4	72	51	39	9	X_4'	72	51	42	12
Y_1	12	42	18	12	Y_1'	12	42	30	24
Y_2	12	66	42	12	Y_2'	12	66	54	24
Y_3	12	72	42	6	Y_3'	12	72	66	30
Y_4	72	39	21	12	Y_4'	72	39	27	18
Y_5	72	42	24	12	Y_5'	72	42	30	18
Y_6	120	24	12	6	Y_6'	120	24	18	12

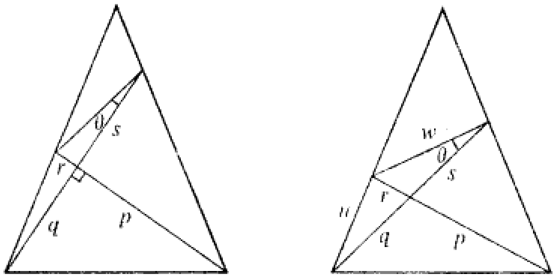
Since Tripp's article appeared, several readers have made further progress with the problem, and we have received contributions from Mr W. J. Courcouf of Westhumble, Surrey; Dr R. Heppinstall of Witham, Essex; Dr J. F. Rigby of University College, Cardiff; and Dr R. F. Wheeler of Leicester University. The account which follows is a summary of progress to date. All of this has been trigonometric in character; no further cases have been established geometrically, nor have any of them been connected together by a 'network' such as that given by Tripp for the triples with $a = 20°$.

The first point to be noticed is that in the four cases X_1 to X_4 the relationship $b + c = 90°$ holds. The trigonometric formula given by Tripp on p. 98 can then be shown to be reducible to the form

$$\tan\theta = \tan(c - \tfrac{1}{2}a)\tan c\tan(c + \tfrac{1}{2}a). \tag{1}$$

This can also be obtained directly from Fig. 2a, using

$$\frac{r}{s} = \frac{r}{q} \times \frac{q}{p} \times \frac{p}{s}. \tag{2}$$



(a) (b)

FIGURE 2.

Equation (1) suggests an argument using roots of equations, and both Rigby and Wheeler made use of this technique. For example, case X_3 requires us to establish that

$$\tan 24° = \tan 6° \tan 42° \tan 78°. \tag{3}$$

To do this, consider the equation

$$\tan 5x = \tan 60°,$$

or

$$\frac{5\tan x - 10\tan^3 x + \tan^5 x}{1 - 10\tan^2 x + 5\tan^4 x} = \sqrt{3}.$$

From the product of the roots of this equation we deduce that

$$\tan 12° \tan 48° \tan 84° \tan 120° \tan 156° = \sqrt{3}.$$

Since $\tan 120° = -\sqrt{3}$, $\tan 156° = -\tan 24°$, $\tan 12° = 1/\tan 78°$, $\tan 48° = 1/\tan 42°$ and $\tan 84° = 1/\tan 6°$, (3) follows at once.

An alternative method is to rewrite (1) in terms of sines and cosines and to use the 'sum and difference formulae' to obtain

$$\{\sin(c + \theta) - \sin(c - \theta)\}\{\cos 2c + \cos a\}$$
$$= \{\sin(c + \theta) + \sin(c - \theta)\}\{\cos a - \cos 2c\},$$

which reduces to

$$\sin(c+\theta)\cos 2c = \sin(c-\theta)\cos a. \tag{4}$$

It is easy to check, with a little further manipulation, that the cases X_1 to X_4 all satisfy (4).

A closer study of Tripp's table (p. 101) shows that, in all the cases which he has solved except one, there is one triplet in each pair for which $b + c = 90°$. (The exceptional pair is (20, 70, 50) and (20, 70, 60).) Equation (4) disposes of all these cases as well. There is, however, an important difference: whereas for these cases the only trigonometric relationships used are $\sin(90° - \alpha) = \cos\alpha$ and $\sin 2\alpha = 2\sin\alpha\cos\alpha$, X_1 to X_4 all reduce to some form of

$$\cos 36° = \tfrac{1}{2} + \sin 18°. \tag{5}$$

Readers will recognise this as a property bound up with the somewhat intricate geometry of the regular pentagon, so it is not surprising that the proofs for these four cases are particularly elusive.

In the Y cases it is necessary to use rather more complicated trigonometric relationships; both Rigby and Heppinstall succeeded in establishing them all. One possible method is to use (2) for Fig. 2b, which leads to

$$\frac{\sin\theta}{\sin(b+c-\theta)} = \frac{\cos(b+\frac{1}{2}a)\sin c\cos(b-\frac{1}{2}a)}{\cos(c-\frac{1}{2}a)\sin b\cos(c+\frac{1}{2}a)} \tag{6}$$

as the generalisation of (1). Heppinstall uses a relationship based on

$$\frac{u}{v} = \frac{u}{w} \times \frac{w}{v} \quad \text{where } v = r + p,$$

which is simpler in having fewer factors although it is less symmetrical than (6). The technique may be illustrated by applying (6) to the triplet Y_1:

$$\begin{aligned}\frac{\sin\theta}{\sin(6\theta° - \theta)} &= \frac{\cos 48°\sin 18°\cos 36°}{\cos 12°\sin 42°\cos 24°}\\ &= \frac{\sin 18°\cos 36°}{\cos 12°\cos 24°}\\ &= \frac{\sin 12°}{\cos 18°} \times \frac{2(2\sin 18°\cos 18°)\cos 36°}{2(2\sin 12°\cos 12°)\cos 24°}\\ &= \frac{\sin 12°\sin 72°}{\cos 18°\sin 48°}\\ &= \frac{\sin 12°}{\sin 48°},\end{aligned}$$

whence the solution $\theta = 12°$ can be 'spotted'. Considerable ingenuity has been shown in some of the solutions, and several of the cases again appear to depend on (5). It is a help to know the answer you are looking for!

There, for the present, the matter rests. Colin Tripp's question (b) is answered. No geometer, however, is likely to be satisfied with the situation as it stands; at least one might hope that some geometrical connection might be found between the various cases with $a = 12°$ and with $a = 72°$. Perhaps this is where other readers may care to take up the challenge.

D.A.Q.

Notes

61.1 Constructing a limaçon locus from the cardioid envelope

It is well known to 'curve-stitchers' that, if equally spaced points around a circle are numbered 1, 2, 3, ..., n (and then repeated cyclically), then the chords joining 1 to 2, 2 to 4, 3 to 6, ..., n to $2n$ envelop a cardioid. (See [1], p. 41, §16, where the cardioid is described as "the caustic of a circle with respect to a point on its circumference".) Correspondence with E. H. Lockwood has led to a rather unexpected result from this envelope.

It is known that if each of the chords PQ in Fig. 1 is extended its own length beyond the initial point P, then the locus of the terminal points is another cardioid. An interesting additional fact is that if each of these same chords is extended its own length in the opposite direction, beyond its extremity Q, then the locus of the terminal points R is a limaçon.

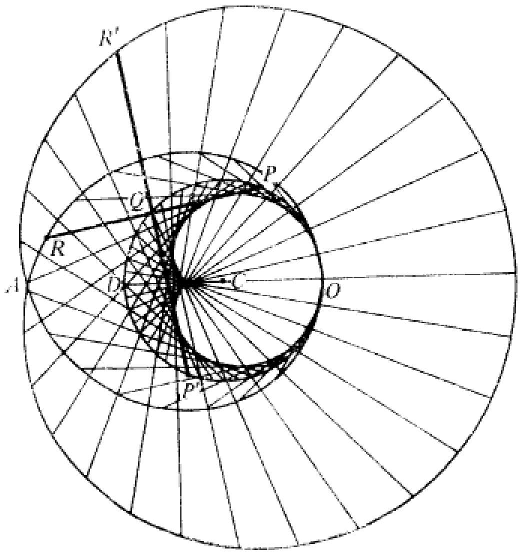


FIGURE 1.