Adventitious angles

COLIN TRIPP

1. Introduction

Many readers will be familiar with the puzzle illustrated in Fig. 1, in which the triangle ABC is isosceles and angles a, b, c are given.† If a, b, c are integers when expressed in degrees then it might appear, at first sight, that so must angle θ be. But this is not the case. The reader will quickly find that θ is not accessible by simple 'angle chasing'. By this I mean marking in the angles obtained by making the angles in any triangle sum to 180° and by making the exterior angle of any triangle equal the sum of the interior opposites. The following question arises: *Given a, b, c can angle θ be obtained by pure geometry?* My conjecture is that the answer is yes in those cases when angles a, b, c and θ are all multiples of the same angle. However, I am far from proving this, and have obtained geometrical derivations only for a limited number of cases, as will be seen.

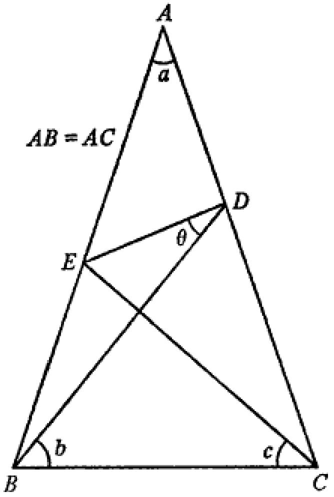

FIGURE 1

2. Trigonometric formula for θ

A formula relating θ to a, b, c can be obtained by repeated application of the sine rule for triangles. This formula may be written

$$\tan\theta = \frac{\sin(b+c)\sin c(\cos a + \cos 2b)}{\sin b(\cos a + \cos 2c) + \cos(b+c)\sin c(\cos a + \cos 2b)}.$$

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† For a recent instance, see the *Bulletin* of the Institute of Mathematics and its Applications for July/August 1974.

Insofar as we are interested in solutions of this equation in which a, b, c and θ are all integers (or all multiples of the same angle), this trigonometric equation may be regarded as a kind of diophantine equation.

3. *Adventitious angles*

DEFINITION 1: With reference to Fig. 1, given that angles a, b, c are multiples of 1°, with $b > c$, the triplet (a, b, c) is *adventitious* if the corresponding angle θ is also a multiple of 1°. We call θ the *derived angle*.

We take $b > c$ to avoid mirror images, since obviously, if (a, b, c) is adventitious, then so is (a, c, b). Also, if $c = b$ we have (a, b, b) which is trivially adventitious ($\theta = b$, by alternate angles between parallels).

4. *An example*

The triplet (20, 60, 50)† is adventitious. However the *proof* that this is so, although elementary, is not obvious. This is one of the cases I have seen set

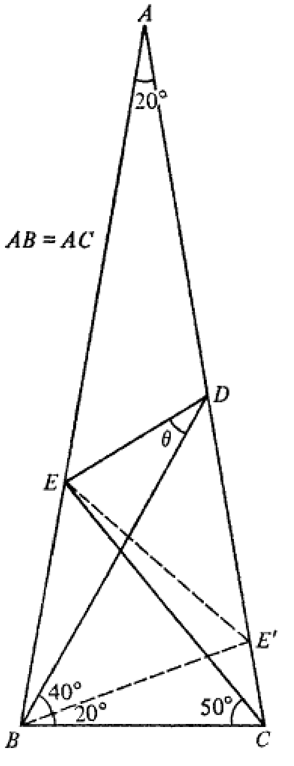

FIGURE 2

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† Strictly (20°, 60°, 50°), but the 'degrees' sign is conveniently omitted.

as a puzzle, and many hundreds of qualified-man-hours must have been spent in trying to derive the value of θ! Substitution in the trigonometric formula for θ, after replacing differences of cosines by products of sines and cancelling $\sin 50°$ top and bottom, gives

$$\tan\theta = \sin 20° \sin 40° \sin 70°/(\sin 60° \sin 30° - \sin^2 20° \sin 40°).$$

It is not enough to look up the various cosines and sines in tables (or use an electronic calculator) because this will only obtain θ to so many decimal place accuracy, and will not prove that θ is an integer. One must reduce the formula still further, making use of trigonometric identities. Considerable ingenuity is required to solve this equation exactly for θ (especially without knowing the solution first!). Rather than follow this through, I give a derivation using only elementary geometry, although a simple construction is needed.

This is to mark a point E' on AC such that angle $E'BC = 20°$ (Fig. 2). It then appears that triangles EBC, $BE'C$ and $DE'B$ are isosceles. Therefore triangle BEE' is equilateral, so triangle $EE'D$ is isosceles. But angle $DE'E = 40°$, so $\theta + 40° = 70°$ giving $\theta = 30°$ exactly. This proves that (20, 60, 50) is adventitious.

5. *Questions which arise*

The above result leads to a number of interesting questions:

(a) Is there any procedure for discovering the construction used in the proof in Section 4?

(b) How many other adventitious triplets are there (among the total of 113 564 possible triplets that can be formed with the restrictions on a, b, c given in the definition in Section 3), and what are they?

(c) Can all the other adventitious triplets be proved to be adventitious by using only pure geometry?

(d) If geometrical proofs exist, and a construction (such as in the example of Section 4) is needed in at least some of the proofs, is there any procedure by which the appropriate construction might be deduced?

I have an answer for (a), a conjecture for (b), a statement of belief for (c) and a partial answer for (d). Questions (a) and (d) will be dealt with in the next section. I have already stated that I believe the answer to (c) to be yes. The conjecture for (b) is based on computer results. A computer search for integer solutions to the trigonometric equation for θ among the 113 564 possible triplets yielded the information that there are 53 triplets which appear to be adventitious to five decimal place accuracy, and these are listed in Table 1. Thus there can be at most 53 adventitious triplets.

If the geometrical figures corresponding to each of these triplets are drawn, it will be noticed that in nearly all cases there appear special geometrical features. Isosceles triangles occur, or lines cross at right angles. In some cases, however, if there are special geometrical features they are not immediately evident. I have proved that 33 of these 53 triplets are actually adventitious, in particular the 8 triplets with $a = 20°$, which are dealt with in the next section.

a	b	c	θ	a	b	c	θ
4	46	4	2	4	46	44	42
8	47	8	4	8	47	43	39
12	42	18	12	12	42	30	24
12	48	12	6	12	48	42	36
12	57	33	15	12	57	42	24
12	66	42	12	12	66	54	24
12	69	21	3	12	69	66	48
12	72	42	6	12	72	66	30
16	49	16	8	16	49	41	33
20	50	20	10	20	50	40	30
20	60	30	10	20	60	50	30
20	65	25	5	20	65	60	40
20	70	50	10	20	70	60	20
24	51	24	12	24	51	39	27
28	52	28	14	28	52	38	24
32	53	32	16	32	53	37	21
36	54	36	18				
40	55	35	15	40	55	40	20
44	56	34	12	44	56	44	22
48	57	33	9	48	57	48	24
52	58	32	6	52	58	52	26
56	59	31	3	56	59	56	28
72	39	21	12	72	39	27	18
72	42	24	12	72	42	30	18
72	48	24	6	72	48	42	24
72	51	39	9	72	51	42	12
120	24	12	6	120	24	18	12

TABLE 1 (angles in degrees)

6. *The eight triplets with* $a = 20°$

Two of the eight of these triplets listed in Table 1 can be proved to be adventitious immediately. The triplet (20, 50, 40) produces a 'kite'-shaped quadrilateral and it follows that $\theta = 30°$ by symmetry. The triplet (20, 50, 20) produces a 'fan' of equal lengths $BC = EC = DC$. An equilateral triangle appears, and it immediately follows that $\theta = 10°$. The other six triplets all require a construction. The fascinating thing is that in each case the required construction can be obtained by superimposing one of the other cases on the figure! For example, for (20, 60, 50) which was proved to be adventitious in Section 4, the construction was obtained by simply superimposing the mirror image of the (20, 50, 20) case on the figure. A network can be established, in which triplets are connected by arrows if the adventitiousness of one triplet can be used to prove the adventitiousness of the other. Fig. 3 illustrates this network. The angle θ is found for (20, 50, 40) and (20, 50, 20) independently as already described, without the use of a construction. These results are used to find θ for (20, 70, 50) and (20, 60, 50), respectively. The proofs that (20, 70, 60) and (20, 60, 30) are adventitious are each based on constructions provided by the (20, 60, 50) figure along with the value of θ for that triplet. The θ value for (20, 60, 30) facilitates the calculation of θ for (20, 65, 60), which in turn enables θ to be calculated for (20, 65, 25). The numbers by the arrows in Fig. 3 indicate the three types of proof that arise:

(1) properties of isosceles (and equilateral) triangles are used;
(2) a cyclic quadrilateral arises, and its properties are used;
(3) two cyclic quadrilaterals arise, and their properties used.

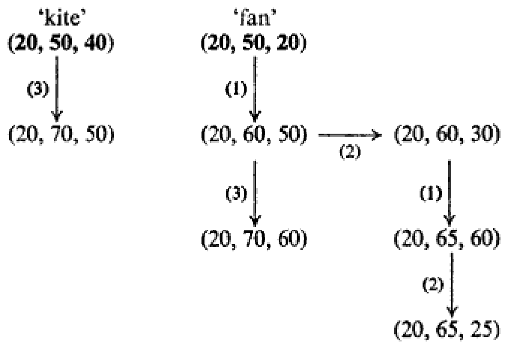

FIGURE 3

It can be seen that (20, 65, 25) seems to be the most inaccessible case, in that at least four other triplets must be proved to be adventitious first. It may, however, be possible to arrive at the adventitiousness of (20, 65, 25) by a shorter route, since the paths shown in the figure are not the only ones possible to establish. For example, it may be shown that the adventitiousness of (20, 70, 50) implies the adventitiousness of (20, 70, 60) by a proof of type (2). I have not established any other paths. It will be noted that all the arrows may be reversed.

It was the cases (20, 60, 50) and (20, 70, 60), set as puzzles, which led to the work described in this article. Taken by themselves, without knowledge of the existence of the other cases, the problem of finding θ is tantalisingly difficult. The reader might like to try one or two of these proofs for himself, in order to appreciate more the elegance and simplicity of each.

7. *Cyclic complements*

In this section the idea that the existence of one adventitious triplet implies the existence of others is developed. It will be noticed from Table 1 that all the triplets fall into pairs (set side by side on the same line), with one exception: (36, 54, 36). In each pair a and b are equal. Theorem 1 explains this, and also why the difference in the values of c in each pair equals the difference in the values of θ.

THEOREM 1. *Given that the set (a, b, c) is adventitious, with derived angle θ, then $(a, b, b-\theta)$ is also adventitious, with derived angle $b-c$.*

PROOF. Construct the circle through EDC; it will intersect AB again at E', between E and B (Fig. 4). In the cyclic quadrilateral $EE'DC$ we have angles $AED = DCE'$ and $ECE' = EDE'$. Now $AED = EBD + EDB$, so these two equations give

$$90° - \tfrac{1}{2}a - b + \theta = 90° - \tfrac{1}{2}a - c' \quad \text{and} \quad c - c' = \theta - \theta',$$

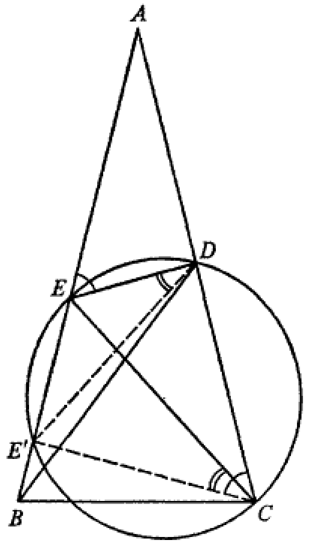

FIGURE 4

where $c' = E'CB$, $\theta' = E'DB$. Thus $c' = b - \theta$ and $\theta' = b - c$. The configuration $AE'BCD$ therefore reveals a second triplet (a, b, c') with derived angle θ', where $c' = b - \theta$, $\theta' = b - c$. It follows that c' and θ' are both integers if b, c and θ are.

This completes the proof. Note that it is the general version of the type (2) proof referred to in Section 6. The other two types of proof may be generalised in the same way, but in these two cases restrictions must be specified on the angles a and b.

The following definition now seems a natural one to make:

DEFINITION 2. Given that the triplet (a, b, c) has derived angle θ, then the triplet $(a, b, b-\theta)$ is its *cyclic complement.*

In Table 1 cyclic complements have been written on the same line. A triplet is its own cyclic complement if $\theta = b - c$. This is found to be the case for (36, 54, 36) (with $\theta = 18°$). I have not proved that there is only one adventitious triplet which is its own cyclic complement.

8. *'Kites' and 'fans'*

It is possible to prove the existence of a set of adventitious triplets independently. These are the ones giving rise to 'fans'. Notice that the cyclic complement of a 'fan' is a 'kite'.

THEOREM 2. *There exist* 27 *adventitious triplets given by the* 14 *triplets* $(a, 45° + \frac{1}{4}a, a)$, *having derived angles* $\theta = \frac{1}{2}a$, *with* $a = 4°, 8°, \ldots, 56°$ *along with their cyclic complements, the triplet with* $a = 36°$ *being its own cyclic complement.*

PROOF. When $b = 45° + \frac{1}{4}a$ and $c = a$, triangles ECB and DCB are isosceles, so that $BC = EC = DC$ (Fig. 5).

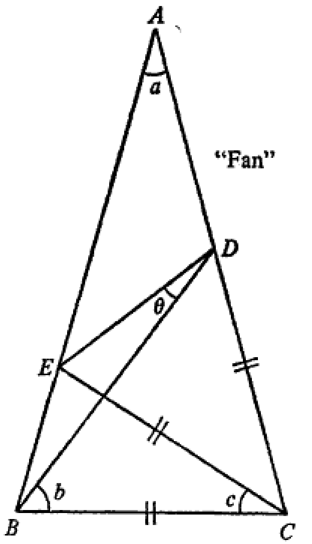

FIGURE 5

Thus triangle EDC is isosceles, so that $\theta = \frac{1}{2}a$. Now $45° + \frac{1}{4}a$ is an integer if a is a multiple of 4, and this in turn ensures that θ is an integer. All angles on the figure must be positive, and in particular, angle $DCE > 0$, i.e. $90° - \frac{3}{2}a > 0$, so that $a < 60°$. This restricts a to the values 4, 8, …, 56. By Theorem 1 each of these 14 triplets has a cyclic complement $(a, 45° + \frac{1}{4}a, 45° - \frac{1}{4}a)$ with derived angle $45° - \frac{3}{4}a$. Only one of these triplets is its own cyclic complement since $45° - \frac{1}{4}a = a$ only for $a = 36°$. This completes the proof.

Two of the 27 triplets proved to be adventitious in Theorem 2 have $a = 20°$, so altogether I have proved that $25 + 8 = 33$ of the 53 triplets suggested by the computer calculations are definitely adventitious. There remain 10 triplets with $a = 12°$, 8 triplets with $a = 72°$ and 2 triplets with $a = 120°$ that still have to be proved adventitious!

9. Generalisations

The idea of adventitious angles may be generalised in two ways. Firstly, there is no reason why a basic unit other than the degree ($= \pi/180$ radians) should not be used.

DEFINITION 3. Given that a, b, c, are each multiples of π/N radians, where N is a positive integer and $b > c$, the triplet (a, b, c) is *N-adventitious* if the corresponding angle θ is a multiple of π/N.

Table 2 gives the number of N-adventitious triplets for various values of N, suggested by computer calculations.

N	Total number of triplets $\frac{1}{6}(\frac{1}{2}N-1)(\frac{1}{2}N-2)(\frac{1}{2}N-3)$	Conjectured number of N-adventitious triplets
14	20	2
16	35	2
24	165	4
42	1 140	10
70	5 984	11
120	32 509	39
168	91 881	34
180	113 564	53
210	182 104	49

TABLE 2

Secondly, the condition that triangle ABC is isosceles may be dropped. This introduces another degree of freedom. The problem really only concerns the quadrilateral $DCBE$ (see Fig. 6).

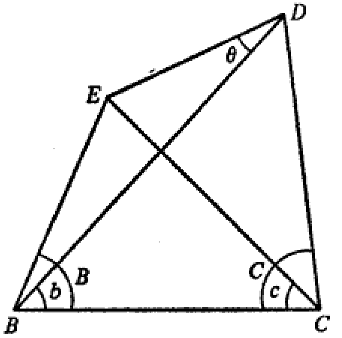

FIGURE 6

DEFINITION 4. With reference to Fig. 6, given that the angles B, b, C, c are multiples of 1°, the quadruplet (B, b, C, c) is *adventitious* if the derived angle θ is also a multiple of 1°.

It is possible that these quadruplets provide a key to proving some of the triplets adventitious.

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Prime factors and recurring duodecimals

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In 1951 I received a letter from Prof. H. Davenport, F.R.S., with some information about the factors of $12^n \pm 1$. The results he gave were interesting and not obvious; whether they were novel I do not know. It occurs to me (whether strictly novel or not) they might well be worth publishing, as of general interest and as a tribute to a great and kindly man. However, since Prof. Davenport's letter was not intended for publication as written, I paraphrase it here with explanation of the background.

Numbers of the form $12^n \pm 1$ can be divided into 4 classes:

$$\text{(A) } 12^{2k}-1, \quad \text{(B) } 12^{2k}+1, \quad \text{(C) } 12^{2k+1}-1, \quad \text{(D) } 12^{2k+1}+1.$$

Now $12^{2k}-1=(12^k+1)(12^k-1)$; by using this identity (repeatedly if necessary) we can factorise numbers of class (A) into those of forms (B), (C), or (D). George S. Terry, of Hingham, Massachusetts, had written to me saying (among other things) that he had found empirically that all factors of (B) numbers were congruent to 1 or 5 (mod 12); similarly all factors of (C) numbers $\equiv 1$ or -1, and factors of (D) numbers $\equiv 1$ or -5. I had written to Davenport asking whether this was true in general, and he replied: "The results you mention are all simple examples of the rule for the quadratic character of 3 to a prime modulus."

Davenport observed that it is sufficient to prove the results for prime factors of $(12^n \pm 1)$, since if these are true the results for composite factors follow immediately. For, if we can show in case (B) that all prime factors $p_1, p_2, p_3, \ldots$ are congruent to 1 or 5 (mod 12), the same holds for any product $p_1p_2p_3\ldots$. Similarly for cases (C), (D). He then appealed to the theory of quadratic reciprocity. The results needed are simply and elegantly explained and proved in Chapter 3 of Davenport's own book [1].

We recall that an integer x is said to be *quadratic* (modulo p) (or a *quadratic residue*) when it is a perfect square in arithmetic modulo p, that is