ON A THEOREM IN GEOMETRY

DANIEL PEDOE, University of Minnesota

If four circles in a plane touch each other externally, and if ϵ_1, ϵ_2, ϵ_3, and ϵ_4 denote their curvatures (that is, the reciprocals of their radii), then the following relation holds between them:

$$2(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2) = (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)^2.$$

A statement of this theorem, without proof, but with a descriptive poem attached, appeared in *Nature*, an English scientific journal, in 1936 [**12**]. The author was Frederick Soddy, distinguished by his pioneering work on isotopes. A year later a generalization of the theorem to $n+2$ spheres in mutual contact, lying in Euclidean space of n dimensions, was published by Thorold Gossett in the same journal [**7**]. Again, no proof was given, but an extra verse for the Soddy poem was suggested.

The title of the Soddy poem, "The kiss precise," is probably enough to indicate its flavour. Coxeter [**3** p. 15] quotes a complete verse.

(It is perhaps worth noting, as a warning to aspiring poetasters, that *Nature* is averse to *belles lettres* these days. A few years ago Sir Alexander Oppenheim submitted the statement of an attractive theorem in geometry, accompanied by a graceful descriptive poem which my wife had written at his suggestion, but *Nature* rejected the joint offering.)

Two proofs of what, for the moment, we shall call the Soddy theorem have appeared in textbooks. One is elementary, uses trigonometry, and does not lend itself to a possible generalization [Coxeter, **3**, p. 15]. The second is apparently simple, but it is not elementary, and uses the methods of the Grassmann calculus as applied to circles by Homersham Cox in 1883 [**2**]. But this proof in Forder's book [**6**] can be generalized. We shall return to it later. There is no reference to Soddy in the Forder proof, and in fact the theorem just occurs as one of a number of worked examples, on p. 339.

The Soddy theorem is an interesting one because it can be viewed from a number of different mathematical angles. We approach it heuristically in the first instance, using arguments which may need amplification, but which do suggest possible lines of attack on the proof.

We do not know *a priori* that a relationship of the Soddy type holds between four circles which touch each other. We do know, from the Apollonius theorem, that eight circles touch three given circles in general position [Pedoe, **11**, p. 23]. When these circles are specialized, some of the eight touching circles may coincide, and, as in all enumerative problems, the coincident solutions of the problem must be counted with a certain multiplicity [Hodge and Pedoe, **9**, vol. II, p. 127].

Suppose now that the three given circles C_1, C_2, C_3 all touch each other. Then since C_1 touches itself, it is a solution of the Apollonius problem. So is C_2, and so is C_3. The multiplicity with which each of these circles must be counted in the solution of the Apollonius problem is (a) greater than 1, and (b) the same for each circle. Let it be n. Then since $3n$ must not exceed 8, we must have $n=2$, and therefore we expect only two other circles to touch three circles which touch each other.

If there is an algebraic relation connecting the curvatures of four circles which touch each other, the above reasoning shows that it must be in the form of a quadratic polynomial equated to zero. In fact, it must be a quadratic *form*, since if we multiply the curvatures of three of the circles by a factor k, by a similarity transformation, the curvature of the fourth circle increases likewise by a factor k. Finally, the form must remain unchanged if we interchange any two of ϵ_1, ϵ_2, ϵ_3, ϵ_4, the curvatures of the four circles. It must therefore equal

$$p\ (\sum \epsilon_i^2) + q \sum \epsilon_i\epsilon_j \quad (1 \leqq i < j \leqq 4).$$

Now, if two circles touch each other externally, and t is the length of a common tangent, we have at once

$$t^2 = (r_1 + r_2)^2 - (r_1 - r_2)^2 = 4r_1r_2,$$

where r_1 and r_2 are the radii. Let three circles of radii r_1, r_2 and r_3 respectively touch each other, and a given line at P, Q and R respectively. Then we have

$$PQ^2 = 4r_1r_2, \quad QR^2 = 4r_2r_3, \quad \text{and} \quad PR^2 = 4r_1r_3.$$

Since $PQ \pm QR = PR$, this leads to

$$1/\sqrt{r_3} \pm 1/\sqrt{r_1} = 1/\sqrt{r_2},$$

which, in terms of the curvatures is $\sqrt{\epsilon_1} \pm \sqrt{\epsilon_2} \pm \sqrt{\epsilon_3} = 0$. This, in rational form, is

$$\sum \epsilon_i^2 - 2 \sum \epsilon_i\epsilon_j = 0 \quad (1 \leqq i < j \leqq 3).$$

Now, the line has curvature $\epsilon_4 = 0$, and this relationship in the case $\epsilon_4 = 0$ indicates (a) that there is an algebraic relation between the four curvatures ϵ_i $(i = 1, \cdots, 4)$ of four circles which touch each other and (b) it must be

$$\sum \epsilon_i^2 - 2 \sum \epsilon_i\epsilon_j = 0 \quad (1 \leqq i < j \leqq 4),$$

and this is the Soddy theorem.

This method can be generalized, if we use the theorem that there is a polynomial relation between the squares of the mutual distances of $n+2$ points in Euclidean space of n dimensions. In two dimensions the relation for four points P_i $(i = a, b, c, d, 4)$ can be written:

$$\sum (P_aP_b)^2(P_bP_c)^2(P_cP_d)^2 - \sum (P_aP_b)^2(P_bP_c)^2(P_cP_a)^2 - \sum (P_aP_b)^2(P_cP_d)^4 = 0,$$

where the summations are over all permutations of unordered pairs, and the first summation contains 12 terms, the second 4, and the third 6. This relation expresses the fact that the volume of the tetrahedron formed by the points is zero, and can be traced back to L. N. M. Carnot [Guggenheimer, **8**].

We can use this relation to indicate the form of the Soddy theorem for five spheres in mutual contact in Euclidean space of three dimensions. We assume that one sphere is a plane, so that the other four spheres touch this plane, and, as above, using the formula $4(P_aP_b)^{-2}=\epsilon_a\epsilon_b$, which is also true for spheres of curvatures ϵ_a, ϵ_b touching a line at P_a, P_b, we obtain the relation

$$\sum \epsilon_i^2 - \sum \epsilon_i\epsilon_j = 0 \quad (1 \leqq i < j \leqq 4),$$

which indicates that the Soddy relation for 5 spheres in mutual contact in Euclidean space of 3 dimensions is

$$\sum \epsilon_i^2 - \sum \epsilon_i\epsilon_j = 0 \quad (1 \leqq i < j \leqq 5),$$

or

$$3\left(\sum \epsilon_i^2\right) = \left(\sum \epsilon_i\right)^2 \quad (i = 1, \cdots, 5).$$

It is worth noting that if we attempt to generalize the method used for circles to show that two spheres touch each of four spheres (in space of 3 dimensions) which are in mutual contact, we run into a serious snag. We may assume, from algebraic considerations that the number of spheres which touch four spheres in general position in space of 3 dimensions is $2^4=16$. If the four given spheres touch each other, and each counts with multiplicity n in the total count, we have $4n+x=16$, where we know that n exceeds 1, and we should like x to equal 2. This will not work. The reason why this method breaks down is that the Soddy theorem is a theorem of *real* Euclidean space, and we cannot expect an algebraic treatment, which gives complex solutions, always to fit the real case.

If we take four sufficiently small spheres round the vertices of a tetrahedron, they will be touched by a sphere that approximates to the circumsphere of the tetrahedron. Noting that each small sphere can lie inside or outside the approximating circumsphere, we see that there are 16 such Apollonian spheres. If the algebraic details are carried out (Prof. Coxeter has done this), it is found that when the radii of the small spheres increase steadily towards the situation of mutual contact, 6 of the Apollonian spheres cease to be real, and 8 of the remaining 10 coincide in pairs with the 4 given spheres, leaving 2 real spheres which touch the 4 given spheres. We wish to find these without having recourse to the algebraic manipulation.

We recollect that our derivation of the Soddy theorem in the plane used properties of the real line. We accordingly begin our investigations again, using a method that proves the Soddy theorem in the plane, generalizes to n dimensions, and makes essential use of real Euclidean space.

To show that two circles touch three given circles in a plane which already touch each other, we invert, with center of inversion at a point of contact. The inverse figure consists of two parallel lines, touched by a circle. Any circle which touches the two lines must have the same radius as the circle already touching the lines, and there are two such circles which touch the circle already touching, *one on each side* of this circle.

We observe that the quadratic equation

$$p(\sum \epsilon_i^2) + q \sum \epsilon_i\epsilon_j = 0 \quad (1 \leqq i < j \leqq 4),$$

must be such that if $\epsilon_3 = \epsilon_4 = 0$ (the two lines each having curvature zero) the equation is identically satisfied by $\epsilon_1 = \epsilon_2$, since the circles have the same radius. Hence

$$p(2\epsilon_1^2) + q(\epsilon_1^2) = 0;$$

we may take $p=1$ and $q=-2$, and obtain the Soddy theorem.

For spheres in three dimensions, we suppose that we are given four spheres which touch each other, and we wish to find how many spheres touch these four spheres. Inverting with respect to a point of contact of two of the spheres, we obtain two parallel planes, each of which is touched by spheres C_1 and C_2 which touch each other. Any sphere which touches the two planes must have radius equal to that of the equal spheres C_1 and C_2. We can construct two equilateral triangles, of which one side is the join of the centers of C_1 and C_2, the triangles lying in the plane through the centers parallel to the two parallel planes in the figure, and the new vertices, *on either side* of the join of the centers, are centers of spheres which touch the two planes and C_1, C_2.

Again the properties of real Euclidean space have been used, and we deduce that there is a form

$$p(\sum \epsilon_i^2) + q \sum \epsilon_i\epsilon_j = 0 \quad (1 \leqq i < j \leqq 5)$$

connecting the curvatures of five spheres which touch each other in real Euclidean space of three dimensions. Again, this form is to be identically satisfied if we put $\epsilon_4 = \epsilon_5 = 0$, and $\epsilon_1 = \epsilon_2 = \epsilon_3$. This leads to

$$p(3\epsilon_1^2) + q(3\epsilon_1^2) = 0;$$

we may take $p=1$, $q=-1$, and deduce once more that the Soddy theorem in three dimensions is

$$3 \sum \epsilon_i^2 = (\sum \epsilon_i)^2 \quad (i = 1, \cdots, 5).$$

The extension to real Euclidean space of n dimensions is immediate. We assume that we have $n+1$ spheres which touch each other, and we invert with respect to a point of contact of two spheres. The inverse figure consists of two parallel hyperplanes, each of which is touched by $n-1$ spheres which also touch each other. These spheres must all have the same radius, and any sphere which touches the two hyperplanes also has this radius. If such a sphere also touches the $n-1$ equal spheres, its center must be at one of two points, and these points are mirror images of each other in the flat space of $n-2$ dimensions spanned by the centers of the $n-1$ spheres. Each of the two centers completes, with the $n-1$ centers, a regular simplex in the unique $(n-1)$-dimensional space which contains the $n-1$ centers and is parallel to the two hyperplanes.

If the curvatures of the $n+2$ spheres in real Euclidean n-space are $\epsilon_1, \cdots, \epsilon_{n+2}$, then if all these spheres touch each other, we have a relation

$$p(\sum \epsilon_i^2) + q(\sum \epsilon_i\epsilon_j) = 0, \quad (1 \leqq i < j \leqq n+2).$$

If we put $\epsilon_{n+1} = \epsilon_{n+2} = 0$, this relation is identically satisfied by $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_n$, so that

$$pn\epsilon_1^2 + q\binom{n}{2}\epsilon_1^2 = 0,$$

and this is satisfied by taking $p = n-1$, and $q = -2$, so that the statement of the Soddy theorem for n dimensions is

$$(n-1)(\sum \epsilon_1^2) - 2(\sum \epsilon_i\epsilon_j) = 0, \quad (1 \leqq i < j \leqq n+2)$$

or

$$n(\sum \epsilon_i^2) = (\sum \epsilon_i)^2 \quad (i = 1, \cdots, n+2).$$

This is the generalization given by Gossett without proof [7].

It was at this point in my investigations that I received a postcard from Prof. Makowski of Warsaw, Poland. He had seen a short reference to my work in *Mathematical Notices*, and his card drew my attention to a paper in Danish by David Fog entitled "The position of a point with reference to certain regular point sets" [5]. This paper was inspired by a paper by M. E. Wise [14] entitled "On the radii of five packed spheres in mutual contact." While working for a Pneumoconiosis Research Unit in Penarth, Wales, Wise had investigated the radii of five spheres in mutual contact, and had discovered the Soddy theorem in three dimensions without knowing about the theorem in two dimensions!

Wise uses inversion, and so does Fog, and as far as this goes both procedures are exactly the same as my own. But Fog then obtains the Soddy theorem in n dimensions by obtaining the polynomial relation which connects the squares of the distances of $n+2$ points in real Euclidean space of n dimensions in the special case when $n+1$ of the points form a regular simplex. Our previous work shows how a regular simplex enters on the scene. Both the Wise and Fog papers are excellent. Let us see how inversion produces the original Soddy theorem in two dimensions without any assumptions about polynomial relations. The proof will illustrate the mechanism underlying the Wise proof for three dimensions.

We have four circles in mutual contact, touching externally, and we are going to invert the figure, choosing the center of inversion at the point of contact of two of the circles. With this point as origin of coordinates, let the equations of the two circles be

$$(x - 1/\epsilon_1)^2 + y^2 - 1/\epsilon_1^2 = 0,$$

and

$$(x + 1/\epsilon_2)^2 + y^2 - 1/\epsilon_2^2 = 0.$$

Take the circle of inversion as

$$x^2 + y^2 - 1 = 0.$$

The inverse of each circle is merely the chord of intersection with the circle of inversion, so that the inverses are

$$x = \epsilon_1/2 \qquad \text{and} \qquad x = -\epsilon_2/2,$$

two parallel lines. The remaining two circles invert into two circles which touch these lines and each other. Their centers must lie on the line $x=(\epsilon_1-\epsilon_2)/4=p$, say, and the radius of each of these circles is

$$r = (\epsilon_1 + \epsilon_2)/4.$$

If one of these circles is $(x-p)^2+(y-q)^2-r^2=0$, it is the inverse of the circle

$$x^2 + y^2 - \frac{2(px + qy)}{p^2 + q^2 - r^2} + \frac{1}{(p^2 + q^2 - r^2)^2} = 0,$$

so that

$$\frac{1}{\epsilon_3^2} = \frac{p^2 + q^2}{(p^2 + q^2 - r^2)^2} - \frac{1}{(p^2 + q^2 - r^2)} = \frac{r^2}{(p^2 + q^2 - r^2)^2},$$

and therefore

$$\pm r\epsilon_3 = p^2 + q^2 - r^2.$$

Since $p^2+q^2-r^2=-\epsilon_1\epsilon_2/4+q^2$, we assume that q is sufficiently large for r to be positive. We now have

$$r\epsilon_3 = p^2 + q^2 - r^2, \tag{1}$$

and therefore also

$$r\epsilon_4 = p^2 + (q + 2r)^2 - r^2, \tag{2}$$

since the center of the fourth *inverted* circle is at a distance $2r$ from that of the third inverted circle, touching it externally.

All we have to do now is to eliminate q from (1) and (2) to obtain the Soddy theorem. We note, however, that (1) is merely a statement of what should be a well-known formula in inversion:

(3) $$(R/R')^2 = (D^2 - R^2)^2,$$

which gives the radius R' of the inverse of a circle of radius R in a unit circle, D being the distance of the center of inversion from the center of the circle of radius R. Using this formula, (1) and (2) are obtained immediately.

To eliminate q, subtract (2) from (1), obtaining

$$r(\epsilon_3 - \epsilon_4) = -4qr - 4r^2;$$

substituting for q^2 in (1) gives

$$16(r\epsilon_3 + r^2 - p^2) = 16q^2 = (4r + \epsilon_3 - \epsilon_4)^2;$$

the substitution of $p = (\epsilon_1 - \epsilon_2)/4$ and $r = (\epsilon_1 + \epsilon_2)/4$ in this equation leads directly to the Soddy theorem. This is probably the simplest elementary proof for the case $n = 2$.

For the case $n = 3$, we should find after inversion that we need a relation between the distances of a given point from the vertices of an equilateral triangle of given side, the point being at a given distance from the plane of the triangle, and so on. This is the method of Wise, extended to n dimensions by Fog.

The assertion we have not yet justified is that there exists an algebraic relation between the curvatures of four circles in a plane which touch each other. This is derived immediately from the identical relation which we have already used connecting the lengths d_{ij} of the six joins P_iP_j of the four centers P_i $(i = 1, \cdots, 4)$ if we put

$$d_{ij} = r_i + r_j,$$

assuming the circles all touch each other externally. The identical relation occurs most naturally in determinantal form as

$$\begin{vmatrix} 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 & 1 \\ d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 & 1 \\ d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 & 1 \\ d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 & 1 \end{vmatrix} = 0,$$

(see, for instance, Kowalesky [10], p. 242). By simple manipulation this determinant can be reduced to the form

$$\begin{vmatrix} -1 & 1 & 1 & 1 & \epsilon_1 \\ 1 & -1 & 1 & 1 & \epsilon_2 \\ 1 & 1 & -1 & 1 & \epsilon_3 \\ 1 & 1 & 1 & -1 & \epsilon_4 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 & 0 \end{vmatrix} = 0,$$

where $\epsilon_i = r_i^{-1}$, and the existence of an algebraic relation between the ϵ_i is established. Moreover it is clearly a symmetric quadratic form in the ϵ_i. This method extends immediately to the case of $n+2$ spheres in mutual contact in Euclidean space of n dimensions, leading to the relation

$$\begin{vmatrix} -1 & 1 & 1 & \cdots & 1 & \epsilon_1 \\ 1 & -1 & 1 & \cdots & 1 & \epsilon_2 \\ 1 & 1 & -1 & \cdots & 1 & \epsilon_3 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & 1 & 1 & \cdots & -1 & \epsilon_{n+2} \\ \epsilon_1 & \epsilon_2 & \epsilon_3 & \cdots & \epsilon_{n+2} & 0 \end{vmatrix} = 0,$$

and our previous methods show that the expanded form of this determinant is

$$n \sum \epsilon_i^2 - \left(\sum \epsilon_i\right)^2 = 0 \quad (i = 1, \cdots, n+2).$$

This method of showing there is an algebraic relation between the ϵ_i is due to the late Prof. A. Aeppli [**1**]. His method for obtaining the coefficients in the expansion of the determinant differs from that given above. I am indebted to his son, my colleague Prof. Alfred Aeppli, for drawing my attention to his father's paper, and also to the fact that his father had traced the theorem we have been discussing (for the case $n=2$) back to Descartes! It appears that Descartes had mentioned the theorem in a letter to the Princess Elizabeth (p. 49 of [**4**]). The theorem was also known to Steiner (p. 63 of Vol. 1 of [**13**]). Evidently Soddy revived the theorem, and certainly obtained a lot of publicity for it, but this is hardly enough for a lasting claim to be laid to the theorem, and we shall now refer to the two-dimensional case as "The Descartes circle theorem," and to the n-dimensional case as "The extension of the Descartes circle theorem." If the theorem is to be named after everyone who has rediscovered it, we should add the names of Steiner, Wise and others. Perhaps one day a Mathematical Council will meet and decide on such matters!

We now approach the configuration of four circles in mutual contact in a plane from another point of view. We represent the circles by points in Euclidean space of three dimensions, E_3. This representation is fully worked out in my book [**11**]. We give a few details here since the same representation will be of use in our next look at the Descartes circle theorem, using the Grassmann calculus.

A normalized circle, with coefficient of x^2+y^2 unity,

$$C \equiv x^2 + y^2 - 2\xi x - 2\eta y + \zeta = 0$$

is represented by the point (ξ, η, ζ) in E_3. This gives a mapping, if we use ordinary rectangular cartesian coordinates, with the point (ξ, η, ζ) immediately above the center of the circle, or, less informally, the center of the circle is the orthogonal projection of the representative point onto the (x, y)-plane. Circles of radius zero (called point-circles) are mapped on the points of the paraboloid of revolution

$$\Omega \equiv z - x^2 - y^2 = 0.$$

Circles of imaginary radius are mapped onto points which lie inside Ω, and are treated equally with ordinary circles, which are mapped onto points outside or on Ω. Circles of the coaxal system $\lambda_1 C_1 + \lambda_2 C_2 = 0$ are mapped onto the points of the line in E_3 which joins the points representing C_1 and C_2. The real limiting points of the coaxal system, if they exist, are given by the orthogonal projection of the points of intersection of this line with Ω. If C_1 and C_2 touch, this line *touches* Ω.

If we erect parallels to the z axis at the points of the circle C, these will intersect Ω in a plane section, of equation

$$z - 2\xi x - 2\eta y + \zeta = 0.$$

This is the *polar plane* of the representative point (ξ, η, ζ) with respect to the quadric Ω. Two points (ξ_1, η_1, ζ_1) and (ξ_2, η_2, ζ_2) are conjugate with regard to Ω, so that

$$2\xi_1\xi_2 + 2\eta_1\eta_2 - \zeta_1 - \zeta_2 = 0$$

if and only if the circles C_1 and C_2 which they represent are *orthogonal.* The point circles on the circumference of C are orthogonal to C. Hence we have a representation of circles by plane sections of Ω, or by points, the *poles* of the plane sections with respect to Ω.

Finally, inversion of circles in a fixed circle C in the plane is represented in E_3 by a *harmonic homology* of the points of E_3, the center of the homology being the point which represents C, and the plane of fixed points being the polar of this point with respect to Ω. The points of Ω which lie in pairs on lines through the center of the harmonic homology are interchanged by the mapping, and project into inverse points in C.

Now, four circles in the plane which touch each other are mapped onto the four vertices of a tetrad in E_3, the sides of which all touch Ω. Harmonic homology applied to such a tetrad produces a tetrad with similar properties in relation to Ω, and there must be some invariant of such a tetrad under a harmonic homology which is the expression of the Descartes theorem for circles in a plane. The discovery of such an invariant would perhaps produce the most satisfactory proof of the Descartes theorem, but, failing this, let us use this theorem to produce a theorem for a tetrad whose six edges all touch a paraboloid of revolution!

We note that if the points A, B of E_3 represent the circles C_1, C_2 of radii ϵ_1^{-1}, ϵ_2^{-1}, which we suppose touch externally at P, then P is the orthogonal projection of the point X on AB where the line AB touches Ω, and we have the simple relation $AX:XB = \epsilon_1^{-1}:\epsilon_2^{-1} = AP:PB$. If we write the Descartes relation as

$$2\left\{1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 + \left(\frac{\epsilon_3}{\epsilon_1}\right)^2 + \left(\frac{\epsilon_4}{\epsilon_1}\right)^2\right\} = \left(1 + \frac{\epsilon_2}{\epsilon_1} + \frac{\epsilon_3}{\epsilon_1} + \frac{\epsilon_4}{\epsilon_1}\right)^2,$$

and A, B, C and D represent the four circles, where AB touches Ω at X, AC touches Ω at Y and AD touches Ω at Z, using the relations $AX:XB=\epsilon_1^{-1}:\epsilon_2^{-1}$, $AY:YC=\epsilon_1^{-1}:\epsilon_3^{-1}$, and $AZ:ZD=\epsilon_1^{-1}:\epsilon_4^{-1}$, we obtain the relation:

$$2\left\{1+\left(\frac{AX}{XB}\right)^2+\left(\frac{AY}{YC}\right)^2+\left(\frac{AZ}{ZD}\right)^2\right\} = \left(1+\frac{AX}{XB}+\frac{AY}{YC}+\frac{AZ}{ZD}\right)^2,$$

connecting the vertices A, B, C, and D and the points of contact X, Y and Z of AB, AC and AD respectively of a tetrad $ABCD$ *all* of whose six edges touch a paraboloid of revolution.

If we had some simple way of proving *this* theorem, the Descartes circle theorem would follow immediately!

Our last look at the Descartes circle theorem is via the Grassmann calculus. The methods of Grassmann were extended and applied to circles by Homersham Cox in 1883 [2], and are given by Forder in his interesting book [6] on the Grassmann calculus. These methods have always smacked of the theologio-mathematical, since higher entities are invoked from time to time, and these can be persuaded to work apparent miracles! But as far as the applications lead to a simple proof of the Descartes theorem, we believe we can explain the calculus in terms of our representation of circles given in the previous section. The same methods will then lead to the generalization of the theorem to n dimensions.

We have seen how circles can be combined linearly, using the mapping:

$$C \equiv x^2 + y^2 - 2\xi x - 2\eta y + \zeta = 0 \rightarrow (\xi, \eta, \zeta).$$

We can also map a circle C on its plane section representation:

$$C \equiv x^2 + y^2 - 2\xi x - 2\eta y + \zeta = 0 \rightarrow z - 2\xi x - 2\eta y + \zeta = 0,$$

and in the present context this is the more convenient mapping. These plane sections can also be combined linearly, but we note that since our circles are normalized, we are essentially using barycentric coordinates in E_3, and we can therefore find *four* linearly independent circles, but any set of five circles is linearly dependent. For instance, although we have not yet mentioned improper circles (of infinite radius), it is clear that the four circles $x=0$, $y=0$, $x^2+y^2-1=0$ and $x^2+y^2+1=0$ are linearly independent, and that any given circle can be expressed linearly in terms of this set of four circles.

There is no difficulty with improper circles. If we consider

$$C_1 - C_2 \equiv -2(\xi_1 - \xi_2)x - 2(\eta_1 - \eta_2)y + \zeta_1 - \zeta_2 = 0,$$

this line, the radical axis of the two circles, is represented without difficulty by the plane section of Ω which has the same equation, the plane being parallel to the z-axis. The point representation in this case breaks down. But if we consider the radical axis of two concentric circles, both representations break down. If the common center is (ξ_1, η_1),

$$C_1 - C_2 \equiv \zeta_1 - \zeta_2 = 0,$$

and there is no finite plane section representation. But we are accustomed to the paradoxical equation: *constant* = 0 representing the plane at infinity, and we introduce this special plane section of Ω, and call it θ. We shall normalize it in a moment.

We observe that θ can always be used as one of the plane sections in E_3 which form a linearly dependent set with any given four planes. In other words, given any four circles, there always exists a linear relation between them of the form

$$x_1C_1 + x_2C_2 + x_3C_3 + x_4C_4 = \theta,$$

where θ is merely some definite non-zero constant.

We now introduce an *inner product* for two circles C_1, C_2. We write

$$[C_1 \cdot C_2] = \tfrac{1}{2}(2\xi_1\xi_2 + 2\eta_1\eta_2 - \zeta_1 - \zeta_2).$$

This looks familiar, and of course it arises very naturally from our point of view. Cox introduces it as

$$[C_1 \cdot C_2] = \tfrac{1}{2}\{r_1^2 + r_2^2 - (O_1 - O_2)^2\},$$

where r_1, r_2 are the respective radii, O_1 and O_2 the respective centers of the two circles. This is an equivalent expression (see [11], p. 32).

The inner product is bilinear and commutative. (This is true for normalized circles; some detail has been omitted here.) Hence its usefulness. It is zero for two orthogonal circles, and $[C_1 \cdot C_1]$, which we write C_1^2, is r_1^2 (hence the introduction of the factor 1/2). If C_1, C_2 touch externally

$$[C_1 \cdot C_2] = \tfrac{1}{2}(r_1^2 + r_2^2 - (r_1 + r_2)^2) = - r_1r_2,$$

and if C_1, C_2 are concentric, $[C_1 \cdot C_2] = (r_1^2 + r_2^2)/2$.

We have one more piece of machinery to set up, and then we can prove our theorem and its generalization. We investigate the inner product of θ with proper circles and with itself. If C_1 and C_2 are concentric,

$$C_1 - C_2 = \zeta_1 - \zeta_2 = k\theta,$$

where k has to be determined. This can be written

$$C_1 - C_2 = (\xi_1^2 + \eta_1^2 - \zeta_2) - (\xi_1^2 + \eta_1^2 - \zeta_1)$$

and we choose $k = (r_1^2 - r_2^2)/2$, thus normalizing θ as the constant -2. We now have $C_1 - C_2 = \frac{1}{2}(r_1^2 - r_2^2)\theta$. Hence

$$[C_1 \cdot (C_1 - C_2)] = [C_1 \cdot (\tfrac{1}{2}(r_1^2 - r_2^2)\theta)] = C_1^2 - [C_1 \cdot C_2]$$
$$= r_1^2 - (r_1^2 + r_2^2)/2 = (r_1^2 - r_2^2)/2.$$

Since $[C_1\cdot(\frac{1}{2}(r_1^2-r_2^2)\theta)]=\frac{1}{2}(r_1^2-r_2^2)[C_1\cdot\theta]$, we deduce that

$$[C_1\cdot\theta] = 1$$

for any proper circle C_1. Finally, since

$$\begin{aligned}\tfrac{1}{4}(r_1^2 - r_2^2)^2\theta^2 &= [(C_1 - C_2)\cdot(C_1 - C_2)]\\ &= C_1^2 - 2[C_1\cdot C_2] + C_2^2\\ &= r_1^2 - (r_1^2 + r_2^2) + r_2^2 = 0,\end{aligned}$$

we find that $\theta^2=0$.

We reassure ourselves about this result by remembering that θ represents a special plane section of Ω, by the plane at infinity, and this touches Ω at infinity, Ω being a paraboloid, and the point of contact, being on Ω, represents a zero circle, so that $C_1^2=r_1^2$ would lead us to expect that $\theta^2=0$. But we could hardly have proved the property this way!

If we weight our circles in advance by multiplying each of them by its curvature, the reciprocal of its radius, then two circles C_1, C_2 touching externally, have the inner product $[C_1\cdot C_2]=-1$ (instead of $-r_1r_2$), and $[C_1\cdot\theta]=1/r_1$ (instead of 1).

We can now prove the Descartes circle theorem. Take four weighted circles of curvatures ϵ_i $(i=1, \cdots, 4)$ which touch each other externally. We can write, the weighted circles being C_i $(i=1, \cdots, 4)$,

$$x_1C_1 + x_2C_2 + x_3C_3 + x_4C_4 = \theta.$$

Forming the inner product of both sides with C_1, we have

$$x_1C_1^2 + x_2[C_1\cdot C_2] + x_3[C_1\cdot C_3] + x_4[C_1\cdot C_4] = [C_1\cdot\theta],$$

that is $x_1-x_2-x_3-x_4=\epsilon_1$. Similarly we obtain three more equations:

$$-x_1 + x_2 - x_3 - x_4 = \epsilon_2,$$

$$-x_1 - x_2 + x_3 - x_4 = \epsilon_3,$$

$$-x_1 - x_2 - x_3 + x_4 = \epsilon_4.$$

We now form the inner product with θ, and we obtain

$$\epsilon_1x_1 + \epsilon_2x_2 + \epsilon_3x_3 + \epsilon_4x_4 = 0.$$

From the first four equations we find that

$$\sum \epsilon_i^2 = 4\sum x_i^2, \quad \text{and} \quad \sum \epsilon_i\epsilon_j = 4\sum x_ix_j.$$

But, substituting for the ϵ_i from the first four equations in the last we find that

$$\sum x_i^2 - 2\sum x_ix_j = 0,$$

so that we have

$$\sum \epsilon_i^2 - 2\sum \epsilon_i\epsilon_j = 0 \quad (1 \leqq i < j \leqq 4),$$

and this is the Descartes circle theorem.

The method of representing circles by points in E_3 can evidently be extended to the representation of spheres in E_n by points in E_{n+1}. The details correspond exactly, and we can at once say that if we have $n+2$ weighted spheres touching each other externally in E_n, we can write

$$x_1C_1 + x_2C_2 + \cdots + x_{n+2}C_{n+2} = \theta,$$

and forming the inner products we have a set of equations

$$x_1 - x_2 - \cdots - x_{n+2} = \epsilon_1,$$

$$\cdots\cdots\cdots$$

$$-x_1 - x_2 - \cdots + x_{n+2} = \epsilon_{n+2},$$

and the equation

$$\epsilon_1x_1 + \epsilon_2x_2 + \cdots + \epsilon_{n+2}x_{n+2} = 0.$$

The subsequent procedures are the same; merely the coefficients are different. We obtain

$$\sum \epsilon_i^2 = (n+2)\sum x_i^2 + (2n-4)\sum x_ix_j,$$

$$\sum \epsilon_i\epsilon_j = (n+1)(\tfrac{1}{2}n - 1)\sum x_i^2 + (2n+2)\sum x_ix_j,$$

from the first set of equations, and substituting in the final equation for the ϵ_i gives again

$$\sum x_i^2 - 2\sum x_ix_j = 0,$$

so that now we have

$$(n-1)\sum \epsilon_i^2 - 2\sum \epsilon_i\epsilon_j = 2n\sum x_i^2 - 4n\sum x_ix_j = 2n\left(\sum x_i^2 - 2\sum x_ix_j\right) = 0,$$

so that the Descartes theorem in n dimensions is

$$n\sum \epsilon_i^2 = \left(\sum \epsilon_i\right)^2, \qquad (i = 1, \cdots, n+2).$$

In conclusion, it may be said of geometry, as has been said of other more popular and less abstract entities, that it is a many-splendoured thing. In fact, since geometry is not at all easy to define, we might hazard the definition that any part of mathematics which can be regarded interestingly from more than one point of view qualifies to be regarded as geometry!

Part of this paper was written under NSF Grant GE-3848 to the University of Minnesota.

References

1. A. Aeppli, Eine Verallgemeinerung einer Formel von Descartes, Elem. Math., 15 (1960) 9–13.
2. H. Cox, Quarterly Journal, 19 (1883) 74.
3. H. S. M. Coxeter, Introduction to Geometry, Wiley, New York, 1961.
4. R. Descartes, Oeuvres de Descartes par Adam et Tannery, vol. 4, Paris, 1901.
5. D. Fog, The position of a point with reference to certain regular point-sets, Nordisk Mat. Tidskr., 8 (1960) 155–169.
6. H. G. Forder, Calculus of Extension, Cambridge Univ. Press, Cambridge, England, 1941, p. 339.
7. T. Gossett, Nature, 139 (1937) 62.
8. H. Guggenheimer, Elem. Math., 19 (1964) 126.
9. W. V. D. Hodge and D. Pedoe, Methods of Algebraic Geometry, Cambridge Univ. Press, Cambridge, England, 1952.
10. G. Kowaleski, Determinantentheorie, W. de Gruyter, Berlin, 1925, p. 243.
11. D. Pedoe, Circles, Pergamon, London, 1957.
12. F. Soddy, Nature, 137 (1936) 1021.
13. J. Steiner, Gesammelte Werke, Reimer, Berlin, 1881.
14. M. E. Wise, On the radii of five packed spheres in mutual contact, Philips Res. Reports, 15 (1960) 101.

MATRICES WITH SPECIFIED EIGENVALUES AND ASSOCIATED EIGENVECTORS, PROPER OR GENERALIZED

LOUIS BRAND, University of Houston

1. Introduction. We shall consider $n\times n$ matrices with elements in the field of reals and limit our discussion to the case when all eigenvalues are real. If λ_i is an eigenvalue and e_i an associated *proper eigenvector*, $Ae_i=\lambda_ie_i$ or

$$(A-\lambda_iI)e_i=0. \tag{1}$$

We shall regard all members of the equivalence class formed by the scalar multiples of e_i as the same eigenvector.

For a multiple eigenvalue λ_1 various cases arise which depend on the rank of the matrix $A-\lambda_1I$. If λ_1 is k-tuple and $\operatorname{rank}(A-\lambda_1I)=n-1$, there will be a single proper eigenvector e_1 that satisfies (1) and $k-1$ *generalized eigenvectors* $e_2, e_3, \cdots, e_k$ which satisfy the $k-1$ equations

$$(A-\lambda_1I)e_i=e_{i-1}, \qquad i=2,3,\cdots,k. \tag{2}$$

We shall show that these equations are consistent, obtain explicit expressions for $e_1, e_2, \cdots, e_k$, and show that these vectors are linearly independent.

When λ_1 is k-tuple and $\operatorname{rank}(A-\lambda_1I)=n-m$, there are m proper and $k-m$ generalized eigenvectors which belong to λ_1. In the extreme case when rank