19th Turkish Mathematical Olympiad

December 3-4, 2011

1. Let $n \geq 2$ be an integer and $E = \{1, 2, \ldots, n\}$. If $A_1, A_2, \ldots A_k$ are subsets of E and exactly one of $A_i \cap A_j, A'_i \cap A_j A_i \cap A'_j$ and $A'_i \cap A'_j$ is empty for all $1 \leq i < j \leq k$, then determine the maximum possible value of k.

[For $A \subset E$, A' denotes the elements of E which are not included in A.]

2. Let D be a point on the side $[BC]$ of the triangle ABC different from the vertices and E be the midpoint of $[CD]$. The line perpendicular to BC at E intersects the side $[AC]$ at point F satisfying $AF \cdot BC = AC \cdot EC$. Let G be the second point where the circumcircle of the triangle ADC intersects the side $[AB]$. Prove that the tangent line of the circumcircle of the triangle AGF at F is also tangent to the circumcircle of the triangle BGE.

3. Show that

$$\frac{1}{x + y^{20} + z^{11}} + \frac{1}{y + z^{20} + x^{11}} + \frac{1}{z + x^{20} + y^{11}} \leq 1$$

for all positive real numbers x, y, z satisfying $xyz = 1$.

4. Let $a_{n+1} = a_n^3 - 2a_n^2 + 2$ for all $n \geq 1$ and $a_1 = 5$. Prove that if $p \equiv 3 \ (mod\ 4)$ is a prime divisor of $a_{2011} + 1$, then $p = 3$.

5. Let $K(M, N)$ be the collection of the midpoints of the line segments with one endpoint belongs to M and the other endpoint belongs to N where M and N are regular convex polygonal regions in the plane. Determine all pairs (M, N) for which $K(M, N)$ is a regular convex polygonal region.

6. Between any two cities of country A consisting of 2011 cities and country B consisting of 2011 cities there is a unique direct two way flight organized by some airway company. For each given city there are at most 19 different airway companies operating flights related to this city. Determine the maximum possible value of the integer k such that no matter how these flights are arranged there are k cities connected (not necessarily directly) only by the flights of some fixed airway company.

Team Selection Test for IMO 2012

March 24-26, 2011

1. Let $A = \{1, 2, \ldots, 2012\}$, $B = \{1, 2, \ldots, 19\}$ and S be the set of all subsets of A. Determine the number of functions $f : S \to B$ satisfying the condition $f(A_1 \cap A_2) = min\{f(A_1), f(A_2)\}$ for all $A_1, A_2 \in S$.

2. In an acute triangle ABC, let D be point on the side $[BC]$ different than the vertices. Let M_1, M_2, M_3, M_4, M_5 be the midpoints of the line segments $[AD]$, $[AB]$, $[AC]$, $[BD]$, $[CD]$, respectively; O_1, O_2, O_3, O_4 be the circumcenters of the triangles $ABD, ACD, M_1M_2M_4, M_1M_3M_5$, respectively; S and T be the midpoints of the line segments AO_1 and AO_2, respectively. Prove that SO_3O_4T is an isosceles trapezoid.

3. Show that

$$a + b + c + \sqrt{3} \geq 8abc \left(\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} \right)$$

for all positive real numbers a, b, c satisfying $ab + bc + ca \leq 1$.

4. The incircle of a triangle ABC touches the sides $[BC], [CA], [AB]$ at points D, E, F, respectively. The circle passing through point A and touches the line BC at D intersects the line segments $[BF]$ and $[CE]$ at the points K and L, respectively. The line passing through E and parallel to DL and the line passing through F and parallel to DK intersect at the point P. Let R_1, R_2, R_3, R_4 denote the circumradius of the triangles AFD, AED, FPD, EPD, respectively. Prove that $R_1R_4 = R_2R_3$.

5. Find all positive integers n for which an integer that can be written as sum of squares of n integers with each of them is divisible by n, can also be expressed as sum of squares of n integers with none of them is divisible by n.

6. Alice and Bob play a game on a $1 \times m$ board using 2012 cards numbered from 1 through 2012. At each step, Alice chooses a card and Bob places it on an empty square of the board. Bob wins the game when numbers on the cards on the board are in an increasing order after k steps where $1 \leq k \leq 2012$, otherwise Alice wins. Find all pairs (k, m) for which Bob can guarantee to win the game.

7. Let $S_r(n) = 1^r + 2^r + \cdots + n^r$ where r is a rational number and n is a positive integer. Find all triples (a, b, c) where a and b are positive rational numbers and c is a positive integer for which there exist infinitely many positive integers n satisfying $S_a(n) = (S_b(n))^c$.

8. Let A, B, C, A', B', C' be distinct points on the plane satisfying $ABC \cong A'B'C'$ and the point G be the centroid of the triangle ABC. If the circle of center A' passing through

G and the circle of diameter $[AA']$ intersect at point A_1, the circle of center B' passing through G and the circle of diameter $[BB']$ intersect at point B_1, the circle of center C' passing through G and the circle of diameter $[CC']$ intersect at point C_1, show that

$$AA_1^2 + BB_1^2 + CC_1^2 \leq AB^2 + BC^2 + CA^2.$$

9. Let $\mathbf{Z}^+$ denote the set of all positive integers and $\mathbf{P}$ denote the set of all prime numbers. For subsets A and S of $\mathbf{Z}^+$, A is called $S-$*proper* if there exists a positive integer N such that for all $a \in A$ and integer b with $0 \leq b < a$ there exist not necessarily distinct elements $s_1, s_2, \dots, s_n$ of S satisfying the conditions $b \equiv s_1 + s_2 + \cdots + s_n \ (mod\ a)$ and $1 \leq n \leq N$. Find a subset S of $\mathbf{Z}^+$ for which $\mathbf{P}$ is $S-$*proper* but $\mathbf{Z}^+$ is not.

16th Junior Turkish Mathematical Olympiad

December 11, 2011

1. Prove that

$$1 \leq \frac{(x+y)(x^3+y^3)}{(x^2+y^2)^2} \leq \frac{9}{8}$$

for all positive real numbers x and y.

2. In a triangle ABC, $|AB| = |AC|$, D is the midpoint of $[BC]$ and E is the foot of the perpendicular from D to the line AC. Let F be the second point where the line BE intersects the circumcircle of the triangle ABD. If G is the intersection point of the lines DE and AF, then show that $|DG| = |GE|$.

3. Let $m < n$ be positive integers and $p = \dfrac{n^2 + m^2}{\sqrt{n^2 - m^2}}$.

a. Find three pairs of positive integers (m, n) for which p is a prime number.

b. Show that if p is a prime number, then $p \equiv 1 \ (mod\ 8)$.

4. Each student in the class has chosen one mathematics and one physics problem out of 20 mathematics and 11 physics problems such that different students choose different pairs of problems. Given that for each student, at least one of the problems chosen by him is chosen by at most one more student, determine the maximum possible number of students in the class.

Team Selection Test for JBMO 2012

May 27-28, 2011

1. Find the largest positive integer n which is divisible by all positive integers whose cube is not greater than n.

2. Find the number of partitions of $\{1, 2, \ldots, 2012\}$ into two sets such that none of the sets contains two distinct elements whose sum is a power of 2.

3. Let the line segment $[AB]$ be a chord of the circle Γ not passing through the center of it and M be the midpoint of $[AB]$. Let C be a variable point on Γ different from A and B, and let P be the point where the tangent line to the circumcircle of the triangle CAM at the point A meets the tangent line to the circumcircle of the triangle CBM at the point B. Show that the lines CP pass through a common fixed point as C varies.

4. Find the greatest constant M such that

$$a^2 + b^2 + c^2 + 3abc \geq M(ab + bc + ca)$$

for all nonnegative real numbers a, b, c satisfying $a + b + c = 4$.

5. Let a, b, c be the lengths of the sides of a triangle; r_a, r_b, r_c be the corresponding exradiuses, respectively, and r be the inradius. Prove that

$$\frac{a+b+c}{2\sqrt{a^2+b^2+c^2}} \leq \frac{\sqrt{r_a^2+r_b^2+r_c^2}}{r_a+r_b+r_c-3r}.$$

6. Determine all positive integers m, n and prime numbers p such that

$$\frac{5^m + 2^n p}{5^m - 2^n p}$$

is a square of an integer.

7. Show that

$$(x^2 + y^2)^3 \geq 32(x^3 + y^3)(xy - x - y)$$

for all real numbers x, y satisfying $x + y \geq 0$.

8. Graph Air (GA) is running two way flights between some cities of a country so that it is possible to travel between any two cities using GA flights. It turned out that after adding one flight, one may travel between any two cities by using at most 17 GA flights. Determine the maximal possible number (if exists) of GA flights necessary to use for traveling between any two cities of a country before adding the flight.

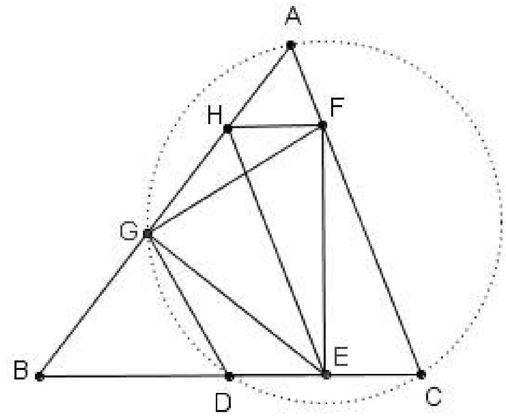
19th Turkish Mathematical Olympiad

Solutions

1. The answer is $2n-3$ and an example is $\{1\},\{2\},\ldots\{n\}$, $\{1,2\}$, $\{1,2,3\}$, $\ldots$ $\{1,2,3,\ldots,n-2\}$.

We will prove it by induction on n. For $n=2$, it is clear that k is at most 1. For $n=3$, it is easy to check that $k\le 3$. Let us assume that the answer is $2n-5$ for $n-1\ge 3$.
Let $M=\{A_1,A_2,\ldots,A_k\}$ be a maximal collection satisfying the conditions for n. By the example above, $k\ge 2n-3$. Note that neither $\emptyset$ nor E is in M. If none of $\{i\}$ and $\{i\}'$ is in M for some $1\le i\le n$, then we could add one of them and enlarge the collection. Clearly both $\{i\}$ and $\{i\}'$ can not be in M and hence exactly one of $\{i\}$ and $\{i\}'$ belongs to M for all $1\le i\le n$.
Observe that if $X\in M$, then we can replace it by X'. Therefore we may assume that $|A_i|\le \dfrac{n}{2}$ for all $1\le i\le n$.
Now let us choose a set $A\in M$ such that $|A|\ge 2$ and $|A|\le |B|$ for all $B\in M$ with $|B|\ge 2$. Since $2n-3>n$, there exists at least one such set. Without loss of generality we may assume that $1,2\in A$. Then consider any set B in M other than $\{1\}$,$\{2\}$ and A.
If $A\cap B=\emptyset$, then $1,2\notin B$.
If $A\cap B'=\emptyset$, then $A\subset B$ and hence $1,2\in B$.
If $A'\cap B=\emptyset$, then $B\subset A$ and hence $|B|=1$ by the choice of A. Thus, $1,2\notin B$.
If $A'\cap B'=\emptyset$, then $A\cup B=E$. But when n is odd $|A|,|B|\le \dfrac{n-1}{2}$ and hence $|A\cup B|\le n-1$. And when n is even, the only possible case is $|A|=|B|=\dfrac{n}{2}$, but then $B=A'$ and $A\cap B=\emptyset$.
Therefore we conclude that $\{1,2\}\subset B$ or $\{1,2\}\cap B=\emptyset$ for all B in M other than $\{1\}$ and $\{2\}$. Hence by removing $\{1\}$ and $\{2\}$ from M and merging 1 and 2, we obtain a new maximal collection for $n-1$. By the induction hypothesis $k-2\le 2n-5$ and we also know that $k\ge 2n-3$. Therefore $k=2n-3$.

2. We will show that EF is a common tangent of the circumcircles of AGF and BGE. Note that it is enough to show that $\angle GBE=\angle GEF$ and $\angle GAF=\angle GFE$.
Let H be the point of intersection of AB and the line passing through F parallel to BC. Then $AHF\sim ABC$ and $\dfrac{AF}{AC}=\dfrac{HF}{BC}$. On the other hand, it is given that $\dfrac{AF}{AC}=\dfrac{EC}{BC}$. Therefore, $HF=EC=ED$ and hence $HFCE$ is a parallelogram and $HFDE$ is a rectangle.
Since $ACDG$ is a cyclic quadrilateral and $FC||HE$, we have $\angle BGD=\angle ACB=\angle HED$. Then H,E,D,G are cyclic. Consequently, H,G,D,E,F are on the circle of diameter $[HE]$. Then $\angle BGE=90^\circ$ and $\angle GBE=90^\circ-\angle GED=\angle GEF$. Since $AGDC$ and $GFED$ are cyclic quadrilaterals $\angle BAC=180^\circ-\angle GDE=\angle GFE$.



3. By the Cauchy-Schwarz inequality we have

$$\frac{1}{a+b^{20}+c^{11}} \leq \frac{a^{13}+b^{-6}+c^{3}}{(a^7+b^7+c^7)^2}$$

for all positive real numbers a, b, c. Summing up the inequalities for $(a,b,c) = (x,y,z),(y,z,x)$ and (z,x,y) gives that it suffices to show that

$$x^{13}+y^{13}+z^{13}+x^{-6}+y^{-6}+z^{-6}+x^3+y^3+z^3 \leq x^{14}+y^{14}+z^{14}+2(x^7y^7+y^7z^7+z^7x^7)$$

to solve the problem.

As $xyz = 1$ we have $x^{13}+y^{13}+z^{13} = \sum_{cyc} x^{13\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$, $x^{-6}+y^{-6}+z^{-6} = \sum_{cyc} x^{6\frac{2}{3}}y^{6\frac{2}{3}}z^{\frac{2}{3}}$ and $x^3+y^3+z^3 = \sum_{cyc} x^{6\frac{2}{3}}y^{3\frac{2}{3}}z^{3\frac{2}{3}}$. Then by Muirhead's inequality we obtain

$$\sum_{cyc} x^{13\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} \leq \sum_{cyc} x^{14}y^0z^0,\ \sum_{cyc} x^{6\frac{2}{3}}y^{6\frac{2}{3}}z^{\frac{2}{3}} \leq \sum_{cyc} x^7y^7z^0 \text{ and } \sum_{cyc} x^{6\frac{2}{3}}y^{3\frac{2}{3}}z^{3\frac{2}{3}} \leq \sum_{cyc} x^7y^7z^0$$

since $(13\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \prec (14,0,0)$, $(6\frac{2}{3}, 6\frac{2}{3}, \frac{2}{3}) \prec (7,7,0)$ and $(6\frac{2}{3}, 3\frac{2}{3}, 3\frac{2}{3}) \prec (7,7,0)$.

4. Observe that $a_{n+1} - 2 = a_n^2(a_n - 2)$ for all $n \geq 1$. By induction on n we obtain $a_{n+1} - 2 = 3a_n^2a_{n-1}^2 \cdots a_1^2$ for all $n \geq 1$. Therefore $a_{2011} + 1 = 3(a_{2010}^2a_{2009}^2 \cdots a_1^2 + 1) = (a_{2010}a_{2009} \cdots a_1)^2 + 1$. Let $p \equiv 3\ (mod\ 4)$ be a prime divisor of $a_{2011} + 1$. It is well known that if q is a prime divisor of $(a_{2010}a_{2009} \cdots a_1)^2 + 1$, then $q \equiv 1\ (mod\ 4)$ or $q = 2$. Thus $p|3$. That is $p = 3$.

5. We will show that $K = K(M,N)$ is a regular polygon if and only if either N is homothetic to M, or N is obtained from M by a $180/m$ degree rotation about the center of M followed by a translation where m is the number of edges of M.

In the following when we say *an edge XY of a convex polygon* it will always be implied that Y is the vertex coming after X counterclockwise along the boundary of the polygon.

Let AB be an edge of M. Then there is a unique line ℓ such that $\ell \cap N$ is either (*1*) an edge $A'B'$ or (*2*) a vertex C of N, and the closed half-planes determined by the lines AB and ℓ and containing M and N, respectively, have nonempty intersection. Let $e(AB)$ denote the central median of the trapezoid $ABB'A'$ in *Case 1* and the midline parallel to the side AB of the triangle ABC in *Case 2*. Let $h(AB)$ denote the closed half-plane defined by the line containing $e(AB)$ which has nonempty intersection with the half-planes mentioned above. The same notation will also be used when the roles of M and N are interchanged.

Let K_1 be the intersection of the half-planes $h(AB)$ for all edges AB of M and N. K_1 is a convex polygon and its boundary is the union of the line segments $e(AB)$ for all edges AB of M and N. Since $h(AB)$ contains K for all edges AB of M or N, K_1 contains K. In fact, $K_1 = K$: If X is in K_1, then (by the Intermediate Value Theorem) X is the midpoint of a line segment YZ where Y and Z are on the edges of K_1. There exists D and E on the edges of M, and D' and E' on the edges of N such that Y and Z are the midpoints of the line segments DD' and EE', respectively. Let F be the midpoint of the line segment DE, and let F' be the midpoints of the line segment $D'E'$. Then F is in M, F' is in N, and X is the midpoint of FF'. This shows that $K_1 = K$.

An edge of K will be called *t-type* in *Case 1* and *M-type* in case *Case 2*. *N-type* is defined similarly. Let d_M and d_N be the edge lengths of M and N, respectively. Then M-type, N-type and t-type edges have lengths $d_M/2$, $d_N/2$ and $(d_M+d_N)/2$, respectively.

Assume that K is a regular polygon. Since K is equilateral, it cannot have M-type and t-type edges or N-type and t-type edges at the same time. If all edges are t-type, then edges of M and N are parallel in pairs, and since M and N are regular polygons, this means that they are homothetic. In this case the reverse implication is obvious.

Now we will consider the remaining case when K is regular: K has both M-type edges and N-type edges, but no t-type edge. Then $d_M = d_N$. Choose adjacent edges of different types: Let AB be an M-type edge of K and let BC be an N-type edge of K. Then $AB = e(A_1B_1)$ is the midline of the triangle $A_1B_1B_2$, and $BC = e(B_2C_2)$ is the midline of the triangle $B_2C_2B_1$ where A_1B_1 is an edge of M and B_2C_2 is an edge of N. Let B_1C_1 be the other edge of M at vertex B_1, and let A_2B_2 be the other edge of N at vertex B_2.

Since $AB//A_1B_1$ and $BC//B_2C_2$, the angle ABC is greater than both the angle $A_1B_1C_1$ and the angle $A_2B_2C_2$. On the other hand, the angle between two adjacent M-type edges of K is equal to an interior angle of M, and the angle between two adjacent N-type edges is equal to an interior angle of N. Since K is regular, we conclude that K cannot have adjacent edges of the same type. Hence M-type and N-type edges must alternate and are equal in number. As the M-type edges are in bijection with the edges of M, and the N-type edges with the edges of N, N is also an m-gon and K is an equilateral $2m$-gon.

Let θ be the acute angle between the lines B_1C_1 and B_2C_2. Then we have the equality $\angle ABC = \angle A_1B_1C_1 + \theta$. This means $180^o \cdot (2m-2)/(2m) = 180^o \cdot (m-2)/m + \theta$. That is, $\theta = 180^o/m$. This completes the proof of the remaining case. In this case the reverse implication follows immediately from the same equality and the relation among the side lengths.

6. The answer is 212.

Let $K_{2011,2011}$ be a complete bipartite graph in which all vertices of a set A with $|A| = 2011$ are connected to all vertices of B with $|B| = 2011$. We prove that there exists a monochromatic connected subgraph with 212 vertices if edges of the graph $K_{2011,2011}$ are colored so that all edges incident to any given vertex are colored by at most 19 colors. Indeed, let i-th color degree of a vertex v be $d_i(v)$. For each connected pair of vertices (u, v), let us define $f(u, v) = d_i(u) + d_i(v)$ where the edge connecting u and v is colored by the i-th color. The set of colors used for coloring of all edges incident to u will be denoted by $C(u)$. The Cauchy-Schwarz inequality implies that

$$\frac{1}{2} \sum_{u \in A, v \in B} f(u, v) = \sum_{u \in A, i \in C(u)} d_i^2(u) \geq \Big(\sum_{u \in A, i \in C(u)} d_i(u) \Big)^2 \cdot \frac{1}{2011 \cdot 19} = 2011^2 \cdot \frac{2011}{19}$$

Therefore, by the pigeon hole principle there exist vertices s, t with $f(s, t) \geq 212$ since $2 \cdot \dfrac{2011}{19} = 211.68$.

Finally, we give an example with the greatest monochromatic connected component of size 212 where only 19 colors are used. Let us partition all vertices of A and B into sets $A_1, \ldots, A_{19}$ and $B_1, \ldots, B_{19}$ off sizes 105 or 106 and color all vertices between A_i and B_j into color $c(i, j)$ where $c(i, j) \equiv i + j \ (mod\ 19)$ and $1 \leq c(i, j) \leq 19$. It can be readily seen that the maximal monochromatic connected component is of size 212.

Team Selection Test for IMO 2012

Solutions

1. The answer is $1^{2012} + 2^{2012} + \cdots + 19^{2012}$.

We first observe that the minimum element of $U \cup V$ is the minimum of the minimum elements of U and V for all finite sets U and V of integers.
Let the value of $f(A)$ be n. Then by the given property of f we have $f(X) \leq n$ for all $X \in S$. Note that there are n^{2012} different ways to determine the values of $f(X)$ for all sets X in S of size 2011. We will prove that the values of f for the remaining sets are uniquely determined.

We apply induction on k to prove that

$$f(A \backslash \{a_1, a_2, \ldots, a_k\}) = min\{f(A \backslash \{a_1\}), f(A \backslash \{a_2\}), \ldots, f(A \backslash \{a_k\})\} \qquad (*)$$

for all $a_1, a_2, \ldots a_k \in B$.
For $k = 1$, it is trivial and the case for $k = 2$ results from the condition on f.

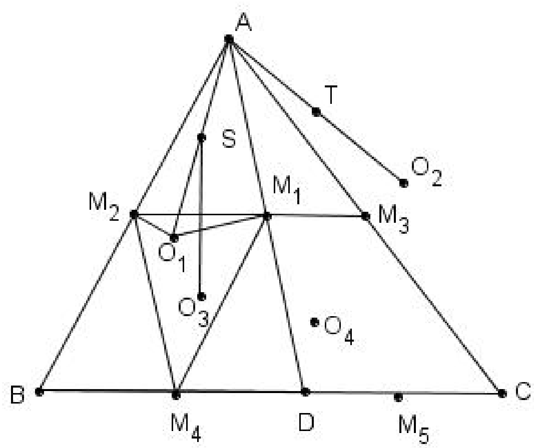
Assume that it is true for $k \geq 2$. By the condition on f, we have

$$f(A\backslash\{a_1, a_2, \dots, a_{k+1}\}) = min\{f(A\backslash\{a_1, a_2, \dots, a_k\}), f(A\backslash\{a_{k+1}\})\}.$$

Using the induction hypothesis and the observation above we can conclude that the claim is true for $k+1$ as well.

On the other hand, by $(*)$ and the observation above it can be easily verified that the condition on f is satisfied for all $A_1, A_2 \in S$. Therefore, as $1 \leq n \leq 19$ we obtain that there are $1^{2012} + 2^{2012} + \cdots + 19^{2012}$ such functions.

2. As $\angle O_1M_1A = \angle O_1M_2A = 90°$, we have that $O_1M_1AM_2$ is a cyclic quadrilateral and the point S is its circumcenter. Hence S is the circumcenter of the triangle AM_1M_2. Next we observe that the triangles AM_1M_2 and $M_4M_2M_1$ are congruent since $AM_1 = M_4M_2 = \dfrac{AD}{2}$ and $AM_2 = M_4M_1 = \dfrac{AB}{2}$. Therefore we obtain that the quadrilateral $SM_1O_3M_2$ is a rhombus and hence the line M_1M_2 is the perpendicular bisector of the line segment $[SO_3]$. Similarly we can get that M_1M_3 is the perpendicular bisector of the line segment $[TO_4]$. As the points M_1, M_2 and M_3 are collinear, the result follows.



3. We first observe that $a^2 + 1 \geq a^2 + ab + bc + ca \geq 4a\sqrt{bc}$ where the second inequality results from $A.M. \geq G.M.$. Therefore we have $2\sqrt{bc} \geq \dfrac{8abc}{a^2+1}$. Summing this up with similar inequalities for b and c gives that it suffices to show that

$$a + b + c + \sqrt{3} \geq 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}).$$

By the Cauchy-Schwarz inequality and $1 \geq ab + bc + ca$, we have

$$\sqrt{3} \geq \sqrt{1+1+1}\sqrt{ab+bc+ca} \geq \sqrt{ab} + \sqrt{bc} + \sqrt{ca}.$$

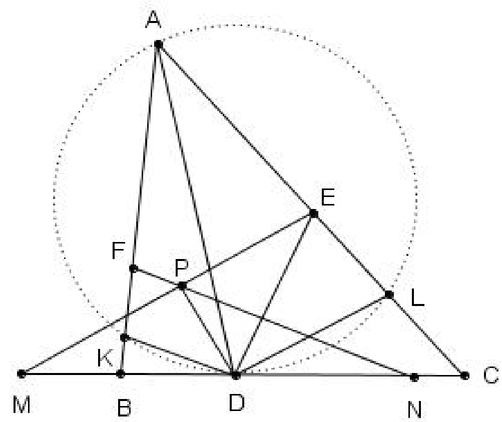
As $(\sqrt{a} - \sqrt{b})^2, (\sqrt{b} - \sqrt{c})^2, (\sqrt{c} - \sqrt{a})^2 \geq 0$ we obtain

$$ab + bc + ca \geq \sqrt{ab} + \sqrt{bc} + \sqrt{ca}$$

and the result follows.

4. Let M be the intersection of the lines PE and BC, N be the intersection of the lines PF and BC. We will prove that $MD = ND$.
The power of B with respect to the circumcircle of the triangle AKD gives $BK \cdot BA = BD^2 = BF^2$, i.e. $BK^2 + BK \cdot KF + BK \cdot AF = (BK + KF)^2$. Therefore, $AF = \dfrac{KF \cdot BD}{BK}$. On the other hand as $FN || KD$ we have $\dfrac{ND}{BD} = \dfrac{KF}{BK}$ and hence $ND = AF$. Similarly we can get $MD = AE$ and then $AE = AF$. Thus, $MD = ND$.

Now note that $\angle DAE = \angle LDC = \angle EMD$ and hence the points A, E, D, M are concyclic. Thus the circumradius of the triangle EMD is R_2. Since $PD = 2R_4 \cdot \sin(\angle PED)$, $MD = 2R_2 \cdot \sin(\angle MED)$ and $\angle MED = \angle PED$, we get $\dfrac{PD}{MD} = \dfrac{R_4}{R_2}$. In a similar way we can get $\dfrac{R_3}{R_1} = \dfrac{PD}{ND}$ and the result follows.



5. The answer is all positive integers except 1,2 and 4.

Let us call a positive integer *good* if it satisfies the condition given in the problem. We first show that if n is good, so is any multiple of n.
Let $m = nk$ and $x_1, x_2, \ldots, x_m$ be integers such that $m|x_i$ for all $1 \le i \le m$. Then since $n|x_i$ for all $1 \le i \le m$ and n is good, there exist integers $y_1, y_2, \ldots, y_m$ such that

$$\sum_{i=nl+1}^{n(l+1)} x_i^2 = \sum_{i=nl+1}^{n(l+1)} y_i^2$$

for all $0 \le l \le k-1$ and $n \nmid y_i$ for all $1 \le i \le m$. Therefore we obtain that

$$\sum_{i=1}^{m=nk} x_i^2 = \sum_{i=1}^{m=nk} y_i^2$$

and $m = nk \nmid y_i$ for all $1 \le i \le m$.

Next we show that all positive odd integers are good.

Lemma: Let n be a positive odd integer and $x_1, x_2, \ldots, x_n$ be integers with at least one of them is not divisibly by n. Then there exist integers $y_1, y_2, \ldots, y_n$ such that none of them is divisible by n and

$$\sum_{i=1}^{n}(nx_i)^2 = \sum_{i=1}^{n} y_i^2.$$

Proof: Without loss of generality we may assume that $n \nmid x_1$. Let $X = 2\sum_{i=1}^{n} x_i$. If $n|X$, then replace x_1 by $-x_1$. As $n \nmid x_1$ and n is odd, $n \nmid 4x_1$ and hence we may assume that $n \nmid X$. Then by the following identity

$$\sum_{i=1}^{n}(nx_i)^2 = \sum_{i=1}^{n}(X - nx_i)^2$$

letting $y_i = X - nx_i$ for all $1 \leq i \leq n$ works.

For a positive odd integer n, if a positive integer a is sum of squares of n integers with each of them is divisible by n, then there exist integers $x_1, x_2, \ldots, x_n$ and a positive integer r such that $a = \sum_{i=1}^{n}(n^r x_i)^2$ and $n \nmid x_i$ for some $1 \leq i \leq n$. Applying the lemma r times we can find integers $y_1, y_2, \ldots, y_n$ such that $a = \sum_{i=1}^{n} y_i^2$ and $n \nmid y_i$ for all $1 \leq i \leq n$.

Next we show that 8 is good. Let a be positive integer which is sum of squares of 8 integers with each of them is divisible by 8. Then $64|a$, hence $a \geq 64$ and $a = 1^2 + 4^2 + 4^2 + 4^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2$ for some integers x_1, x_2, x_3, x_4 by Lagrange's four-square theorem. Note that $x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv 7 \ (mod\ 8)$ and the only way to get 7 as sum of four quadratic residues in $(mod\ 8)$ is $1+1+1+4$. Therefore, $8 \nmid x_i$ for all $1 \leq i \leq 4$.

Finally, we observe that $32 = 4^2 + 4^2 + 0^2 + 0^2$ is a counterexample for 4 and we are done.

6. Answer: Bob wins for all pairs of $(k, m \geq 2^k - 1)$ if $k = 1, 2, \ldots, 10$ and for all pairs $(k, m \geq 2012)$ if $2012 \geq k \geq 11$.

Let us show that Bob wins in all cases listed above. If $k = 1$ then $2^k - 1 = 1$ and the result is trivial. Suppose that Bob wins for $k - 1 \leq 9$ when $m \geq 2^{k-1} - 1$. Bob places the first card on a square numbered $2^k - 1$. The square $2^k - 1$ divides the whole board into two parts. Let L be the number on the first card chosen by Alice. After the first move all cards numbered less than L will be placed to the left part and all cards numbered greater than L will be placed to the right part. Note that both parts having sizes not less than $2^{k-1} - 1$ and by assumption Bob has a winning strategy for remaining $k - 1$ moves and we are done.

If $k \geq 11$ and $m \geq 2012$, then Bob just places the card numbered L to the square numbered L.

Now we show that Alice wins in all remaining cases. Alice's strategy: Let us line the cards in increasing order. Suppose Alice's i-th move is a card numbered L. Bob places the card numbered L into some square. After Bob's $i-1$-th move, the set of all vacant squares of the board $1 \times m$ is naturally decomposed into connected components. The card L divides the connected component I into two parts, say I_{left} and I_{right}. Alice chooses the part I' not exceeding the other one in length. The set of remaining cards is also naturally decomposed into connected components. If the part I' is I_{left} then Alice will choose a card from connected component of remaining cards ending at $L-1$ for the next step, if the part I' is I_{right} then Alice will choose a card from the connected component of remaining cards starting at $L+1$ for the next step. In her $i+1$-th move, if she needs to choose a card from component $[N, M]$ she chooses a card numbered $\lceil (N+M)/2 \rceil$. It can be readily seen that Alice wins in all remaining cases if she starts with a card numbered 1006.

7. The answer is $a = 3, b = 1, c = 2$ and $a = b \in \mathbb{Q}^+, c = 1$.

Applying induction on n by using Bernoulli's inequality gives

$$\frac{n^{r+1}}{r+1} \leq S_r(n) \leq \frac{(n+1)^{r+1}}{r+1}$$

for all positive integer n and positive rational number r.
As $S_a(n) = (S_b(n))^c$ letting $r = a$ and $r = b$ gives that

$$\frac{n^{a+1}}{a+1} \leq (\frac{(n+1)^{b+1}}{b+1})^c \text{ and } (\frac{n^{b+1}}{b+1})^c \leq \frac{(n+1)^{a+1}}{a+1}$$

i.e.

$$\frac{n^{(b+1)c}}{(n+1)^{a+1}} \leq \frac{(b+1)^c}{a+1} \leq \frac{(n+1)^{(b+1)c}}{n^{a+1}}$$

holds for infinitely many positive integers n. By letting $n \to \infty$ in the last inequality, we obtain that $(b+1)c = a+1$ and $(b+1)^c = a+1$. If $c = 1$, then $a = b$ and we get the trivial solutions.
If $c > 1$, then $c = (b+1)^{c-1}$ implies that b is an integer since c is an integer. As $b \geq 1$, we get that $c \geq 2^{c-1}$ and hence $c = 2$. This leads to $b = 1, a = 3$ and this solution clearly satisfies the condition.

8. *Lemma:* Let X, Y, Z, T be points on a plane. Then

$$XY^2 + YZ^2 + YT^2 + TX^2 \geq XZ^2 + YT^2.$$

Proof: Let $x = \overrightarrow{XY}, y = \overrightarrow{YZ}, z = \overrightarrow{ZT}$. Note that $\overrightarrow{XZ} = x + y, \overrightarrow{YT} = y + z$ and $\overrightarrow{XT} = x + y + z$. Then $XY^2 + YZ^2 + YT^2 + TX^2 - XZ^2 - YT^2$ is equal to

$$x \cdot x + y \cdot y + z \cdot z + (x+y+z) \cdot (x+y+z) - (x+y) \cdot (x+y) - (y+z) \cdot (y+z)$$
$$= x \cdot x + z \cdot z + 2(x \cdot z) = (x+z) \cdot (x+z) \geq 0.$$

Let the point G' be the centroid of the triangle $A'B'C'$. Applying the lemma for A, G', A', G gives $AG'^2 + G'A'^2 + A'G^2 + GA^2 \geq G'G^2 + AA'^2$. As $AA'^2 = A'G^2 + AA_1^2$ we have

$$AA_1^2 \leq AG'^2 + G'A'^2 + GA^2 - G'G^2.$$

By similar inequalities for B and C, we see that $AA_1^2 + BB_1^2 + CC_1^2$ is less than or equal to

$$AG'^2 + G'A'^2 + GA^2 + BG'^2 + G'B'^2 + GB^2 + CG'^2 + G'C'^2 + GC^2 - 3G'G^2. \quad (*)$$

By the Leibniz's theorem we obtain that $AG'^2+BG'^2+CG'^2-3G'G^2 = GA^2+GB^2+GC^2$. It is well known that $GA^2+GB^2+GC^2 = \dfrac{1}{3}(AB^2+BC^2+CA^2)$. As $ABC \cong A'B'C'$, we also have $G'A'^2 + G'B'^2 + G'C'^2 = \dfrac{1}{3}(AB^2 + BC^2 + CA^2)$. These three results conclude that $(*)$ is equal to $AB^2 + BC^2 + CA^2$ and we are done.

9. We will show that the set $S = \{k \cdot 5^{k-1} : k \in \mathbf{Z}^+\}$ is an appropriate example.

We first prove that $\mathbf{P}$ is S-proper for $N = 4$. Let $p \neq 5$ be prime and $0 \le n < p$. Then by Chinese remainder theorem there exists a positive integer k satisfying the conditions $k \equiv n\ (mod\ p)$ and $k \equiv 1\ (mod\ p-1)$. Therefore, by Fermat's theorem we obtain that $k \cdot 5^{k-1} \equiv n\ (mod\ p)$. For $p = 5$; as $1, 10 \in S$, $N = 4$ works.

Now suppose that $\mathbf{Z}^+$ is S-proper. Then there exists a positive integer N satisfying the given conditions. Therefore we can get any residue in $(mod\ 5^N)$ as sum of at most N elements of S. Since $k \cdot 5^{k-1} \equiv 0\ (mod\ 5^N)$ for all $k > N$, we have that for each integer $0 \le n < 5^N$, there exist non-negative integers $x_1 x_2, \ldots, x_N$ such that

$$x_1 \cdot 1 + x_2 \cdot 2 \cdot 5 + \cdots + x_i \cdot i \cdot 5^{i-1} + \cdots + x_N \cdot N \cdot 5^{N-1} \equiv n\ (mod\ 5^N)$$

and $x_1 + x_2 + \cdots + x_N \le N$. Observe that for each n, we get a different N-tuple. On the other hand, the number of N-tuples of non-negative integers satisfying $x_1 + x_2 + \cdots + x_N \le N$ is $\binom{2N}{N} < 5^N$ and hence we get a contradiction.

16th Junior Turkish Mathematical Olympiad

Solutions

1. The Cauchy-Schwarz inequality implies that $(x+y)(x^3+y^3) \ge (x^2+y^2)^2$. Therefore,

$$1 \le \frac{(x+y)(x^3+y^3)}{(x^2+y^2)^2}.$$

As $0 \le ((x-y)^2 - 2xy)^2$, we have

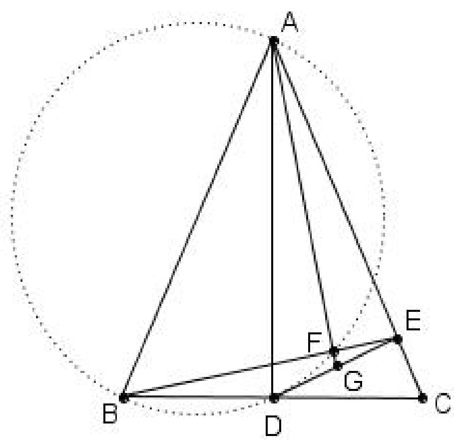
$$4xy(x-y)^2 \le (x-y)^4 + 4x^2y^2.$$

$\Rightarrow 8x^3y + 8y^3x \le x^4 + 18x^2y^2 + y^4$. $\Rightarrow 8x^3y + 8y^3x + 8x^4 + 8y^4 \le 9x^4 + 18x^2y^2 + 9y^4$. $\Rightarrow 8(x+y)(x^3+y^3) \le 9(x^2+y^2)^2$. Thus,

$$\frac{(x+y)(x^3+y^3)}{(x^2+y^2)^2} \le \frac{9}{8}.$$

2. Let $\angle ACB = \alpha$. Since $AB = AC$, we have $\angle ABD = \angle ABC = \alpha$. $DE \perp AC$ and $AD \perp BC$ imply that $\angle EDC = 90° - \alpha$ and $\angle ADE = \alpha$. Therefore, we get $\angle ABD = \angle ADE = \alpha$ which implies that DE is tangent to the circumcircle of the triangle ABD and hence $GD^2 = GF \cdot GA$.

On the other hand, $\angle BFA = \angle ADB = 90°$. Therefore, EF is perpendicular to the hypothenuse of the right triangle AEG. Thus, $GE^2 = GF \cdot GA$ and consequently we get $GD = GE$.



3. a. For $(m, n) = (6, 10), (12, 15)$ and $(30, 78)$, p is equal to

$$\frac{10^2 + 6^2}{8} = 17, \frac{15^2 + 12^2}{9} = 41 \text{ and } \frac{78^2 + 30^2}{72} = 97,$$

respectively.

b. Let $k = \sqrt{n^2 - m^2}$. Then (k, m, n) is a Pythagorean triple. Therefore, there exist positive integers d, x and y such that $(x, y) = 1$, $n = d(x^2 + y^2)$ and $m = d(x^2 - y^2)$ or $m = 2dxy$.

If $m = d(x^2 - y^2)$, then $p = \dfrac{d(x^4 + y^4)}{xy}$. Since $(x, y) = 1$, we have $(xy, x^4 + y^4) = 1$ and hence xy divides d. As $x^4 + y^4 \geq 17$ and p is a prime, we get $d = xy$ and $p = x^4 + y^4$. Since one of x and y is even and the other one is odd, $p = x^4 + y^4 \equiv 1 \ (mod\ 8)$.

If $m = 2dxy$, then $p = d(x^2 - y^2) + \dfrac{8dx^2y^2}{x^2 - y^2}$. Since $\dfrac{8dx^2y^2}{x^2 - y^2}$ is an integer and $(xy, x^2 - y^2) = 1$, we have $x^2 - y^2$ divides $8d$. On the other hand, as p is a prime number, d and $\dfrac{8d}{x^2 - y^2}$ are relatively prime. Thus, $x^2 - y^2$ is a multiple of d. Therefore, $x^2 - y^2 = d, 2d, 4d$ or $8d$.

If $x^2 - y^2 = d$, then as $p = d^2 + 8x^2y^2$ is a prime, d is odd and $p \equiv 1 \ (mod\ 8)$.
If $x^2 - y^2 = 2d$, then $p = 2(d^2 + 2x^2y^2)$ is not a prime number.
If $x^2 - y^2 = 4d$, then $p = 2(2d^2 + x^2y^2)$ is not a prime number.

If $x^2 - y^2 = 8d$, then $p = 8d^2 + x^2y^2$ is a prime number implies that both x and y are odd numbers and consequently $p \equiv 1 \ (mod\ 8)$.

4. The answer is 54.

For $1 \leq i \leq 20$ and $1 \leq j \leq 11$ we define $a_{i,j}$ as follows; $a_{i,j} = 1$ if i-th mathematics problem and j-th physics problem are chosen by some student, $a_{i,j} = 0$ otherwise. Now we can reformulate the problem: Find the maximal possible value of the expression $A = \sum_{i=1}^{20} \sum_{j=1}^{11} a_{i,j}$ under the following two conditions:

• $a_{i,j} = 0$ or 1

• if $a_{k,l} = 1$ for some k and l, then at least one of the sums $\sum_{j=1}^{11} a_{k,j}$ and $\sum_{i=1}^{20} a_{i,l}$ does not exceed 2.

First of all, let us show that $A \leq 54$. Suppose that $a_{k,l} = 1$. We say that k is *1-good*, if $\sum_{j=1}^{11} a_{k,j} \leq 2$; we say that l is *2-good* if $\sum_{i=1}^{20} a_{i,l} \leq 2$.

If the total number of 1-good values of k is 20, then $A \leq 2 \cdot 20 = 40$.
If the total number of 2-good values of l is 11, then $A \leq 2 \cdot 11 = 22$.
If the total number of 1-good values of k is 19, then $A \leq 2 \cdot 19 + 11 = 49$.
If the total number of 2-good values of l is 10, then $A \leq 2 \cdot 11 + 20 = 32$.

Finally, if the total number of 1-good values of k is less than or equal to 18 and the total number of 2-good values of l is less than or equal to 9, then the total number of good values is at most 27 and readily $A \leq 2 \cdot 27 = 54$, since the number of nonzero terms of A is less than or equal to twice the number of good values. Thus, $A \leq 54$.

Now we give an example for $A = 54$. Let $a_{i,j} = 1$ only for

$$(i, j) \in \{(i, j) : i \in \{1, 20\} \text{ or } j \in \{1, 11\}\} \backslash \{(1, 1), (20, 1), (1, 11), (20, 11)\}.$$

The conditions are readily satisfied and we are done.

Team Selection Test for JBMO 2012

Solutions

1. The answer is 420.

Let us consider the positive integer m so that $m^3 \leq n < (m+1)^3$. As $n = 420$ satisfies the conditions, we will consider the case when $m \geq 7$. Note that each of $m, m-1, m-2$ and $m-3$ divides n and hence $lcm(m, m-1, m-2, m-3)$ divides n. Since $gcd(n-1, n-2) = 1$, $gcd(n, n-3)$ divides 3 and $gcd(n(n-3), (n-1)(n-2))$ divides 2, we have that

$\dfrac{m(m-1)(m-2)(m-3)}{6}$ divides n. Therefore, $\dfrac{m(m-1)(m-2)(m-3)}{6} < (m+1)^3$ and hence $m \leq 12$.

If $m = 11$ or 12, then $11 \cdot 7 \cdot 5 \cdot 9 \cdot 8 = 27720|n$, but $13^3 = 2197 < 27720$.
If $m = 9$ or 10, then $7 \cdot 5 \cdot 9 \cdot 8 = 2520|n$, but $11^3 = 1331 < 2520$.
If $m = 8$, then $7 \cdot 5 \cdot 3 \cdot 8 = 840|n$, but $9^3 = 729 < 840$.
If $m = 7$, then $7 \cdot 5 \cdot 3 \cdot 4 = 420|n$ and $n < 8^3 = 512$. Therefore $n = 420$.

2. The answer is 1024.

Let us call a partition of $\{1, 2, \ldots, n\}$ into two sets *nice partition for* n, if none of the sets contains two distinct elements whose sum is a power of 2. Let p_n be the number of nice partitions for n. We observe that removing n from a nice partition for n gives a nice partition for $n-1$. Therefore, we can obtain the nice partitions for n by adding n to the nice partitions for $n-1$.

If $2^m < n < 2^{m+1}$ for some positive integer m, then as $2^m < n+1 \leq n+(n-1) < 2^{n+2}$ and $1 \leq 2^{m+1} - n \leq n-1$, we must add n into the set not containing $2^{m+1}-n$. Therefore, $p_n = p_{n-1}$.

If $n = 2^m$ for some positive integer m, then since $2^m < n+1 \leq n+(n-1) < 2^{m+1}$, we can add n into any of the sets and hence $p_n = 2 \cdot p_{n-1}$.

As $p_2 = 2$ and $2^{10} < 2012 < 2^{11}$, we conclude that $p_{2012} = 2^{10} = 1024$.

3. Let Q be the point where the tangent lines to Γ at A and B meet. We will show that Q is the point.

Since $\angle MCA = \angle MAP$ and $\angle MCB = \angle MBP$, we have $\angle ACB + \angle APB = 180^\circ$ and P lies on Γ. Therefore, if P' is the point where QC intersect the circle, it suffices to show that $\angle MCA = \angle BCP'$ as then it will follow that AP' is tangent to the circumcircle of the triangle CAM and $P' = P$.

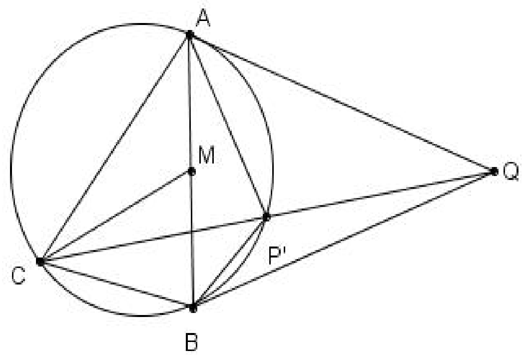
The triangles QAP' and QCA, and the triangles QBP' and QCB are similar. Therefore

$$\frac{AP'}{CA} = \frac{QA}{QC} = \frac{QB}{QC} = \frac{BP'}{CB}.$$

Then by the Ptolemy's Theorem,

$$CP' = \frac{CA \cdot BP' + CB \cdot AP'}{BA} = \frac{CA \cdot BP'}{MA} \text{ and } \frac{CP'}{BP'} = \frac{CA}{MA}.$$

We conclude that the triangles $CP'B$ and CAM are similar, and hence $\angle MCA = \angle BCP'$.



4. Letting $a = 0$ and $b = c = 2$ we obtain $2 \geq M$. We will show that $M = 2$ works.

Without loss of generality we may assume that $\max\{a, b, c\} = c$. Let $x = a + b$ and $y = ab$. We have $c \geq \dfrac{a+b+c}{3} = \dfrac{4}{3}$ and hence $x = a + b \leq \dfrac{8}{3}$.

Then $a^2 + b^2 + c^2 + 3abc \geq 2(ab + bc + ca) \iff x^2 - 2y + (4 - x)^2 + 3y(4 - x) \geq 2(y + x(4 - x)) \iff 4(x - 2)^2 + y(8 - 3x) \geq 0$ follows.

5. Note that

$$\frac{a+b+c}{2}r = \frac{b+c-a}{2}r_a = \frac{a+b+c}{2}r_a - ar_a = \frac{a+b+c}{2}r_b - br_b = \frac{a+b+c}{2}r_c - cr_c.$$

Therefore, we have $\dfrac{3r(a+b+c)}{2} = \dfrac{a+b+c}{2}(r_a + r_b + r_c) - ar_a - br_b - cr_c$ and hence

$$ar_a + br_b + cr_c = \frac{a+b+c}{2}(r_a + r_b + r_c - 3r).$$

On the other hand, the Cauchy-Schwarz inequality implies

$$ar_a + br_b + cr_c \leq \sqrt{a^2 + b^2 + c^2} \cdot \sqrt{r_a^2 + r_b^2 + r_c^2}$$

and the result follows.

6. The answer is $(m, n, p) = (2, 3, 3), (1, 1, 2)$ or $(2, 2, 5)$.

Let $\dfrac{5^m + 2^n p}{5^m - 2^n p} = k^2$ for some positive integer k. Note that $5^m - 2^n p | 5^m + 2^n p$ implies that $5^m - 2^n p | 2 \cdot 5^m$. Then as $5^m - 2^n p$ is odd, $5^m - 2^n p | 5^m$ and hence $5^m - 2^n p = 5^r$ for some non-negative integer r.

Case 1: $r = 0$ i.e. $5^m - 2^n p = 1$.

If $n \geq 3$, then $5^m \equiv 1$ $(mod\ 8)$ and hence $m = 2s$ for some positive integer s. Then $5^{2s} \equiv 1$ $(mod\ 3)$ and we have $2^n p \equiv 0$ $(mod\ 3)$. Thus, $p = 3$ and $(5^s - 1)(5^s + 1) = 3 \cdot 2^n$. Observe that $5^s + 1 \equiv 2$ $(mod\ 4)$ and has an odd divisor greater than 3 when $s > 1$.

Therefore $s = 1$ and hence $m = 2, n = 3$ and $k = 7$.

If $n = 2$, then $8p = (5^m + 2^2p) - (5^m - 2^2p) = k^2 - 1$. Therefore $k = 2l + 1$ for some positive integer l and $2p = l(l + 1)$. Then clearly $p = 3$ and hence $5^m = 13$ which yields a contradiction.

If $n = 1$, then $4p = (5^m + 2^1p) - (5^m - 2^1p) = k^2 - 1$. Therefore $k = 2l + 1$ for some positive integer l and $p = l(l + 1)$. Then clearly $l = 1, p = 2$ and hence $k = 3, m = 1$.

Case 2: $r \geq 1$.

Then $5|2^np$ and hence $p = 5$. Therefore, $5^{m-1} - 2^n = 5^{r-1}$ implies that $r = 1$ since $m > r$ and $5^{r-1}|2^n$. Thus, we have $5^{m-1} - 2^n = 1$. Clearly $n \neq 1$ and if $n = 2$, then $m = 2$ and $k = 3$.

If $n \geq 3$, then $5^{m-1} \equiv 1 \ (mod\ 8)$ and hence $m - 1 = 2s$ for some positive integer s. Then $(5^s - 1)(5^s + 1) = 2^n$. Observe that $5^s + 1 \equiv 2 \ (mod\ 4)$ and has an odd divisor greater than 1 when $s \geq 1$. Therefore $s = 0$ and hence $2^n = 0$ which yields a contradiction.

7. Let $s = x^2 + y^2$ and $t = x + y$. We want to show that

$$s^3 \geq 8t(3s - t^2)(t^2 - 2t - s)$$

for $2s \geq t^2$ and $t \geq 0$.

Now let $s = rt$. This transforms the inequality to

$$r^3 \geq 8(3r - t)(t - 2 - r)$$

for $2r \geq t \geq 0$.

Since $r^3 - 8(3r - t)(t - 2 - r) = 8(t - (2r + 1))^2 + r^3 - 8r^2 + 16r - 8 \geq r^3 - 8r^2 + 16r = r(r - 4)^2 \geq 0$ for $2r \geq t \geq 0$, we are done.

8. The answer is 34.

Let $A_1, A_2, \ldots, A_{35}$ be cities so that only A_i and A_{i+1} are connected for $i = 1, 2, \ldots, 34$. The travel between A_1 and A_{35} uses at least 34 flights. After adding flights between A_1 and A_{35}, it is possible to travel between any pair of cities by using of at most 17 flights. Therefore, the answer is at least 34.

Let us show that 34 flights are sufficient. Let $\rho(X, Y)$ denote the minimal possible number of flights between the cities X and Y. On the contrary, suppose that $\rho(A, B) > 34$ for some cities A and B and a path with minimal number of flights before adding a flight between the cities T and S is $(A = A_0, A_1, \ldots, A_{17}, A_{18}, \ldots, B = A_k)$. After adding the flight between T and S, $\rho(A, A_{18}) \leq 17$ and $\rho(A_{17}, B) \leq 17$. By definitions, both of the paths with minimal number of flights from A to A_{18} and from A_{17} to B have to use the flight between T and S. Without loss of generality we may assume that the path from A to A_{18} is $(A_0, \ldots, T, S, \ldots A_{18})$. Then note that $l_1 + l_2 \leq 16$ where $\rho(A, T) = l_1, \rho(S, A_{18}) = l_2$. Similarly, the path from A_{17} to B is $(A_{17}, \ldots, T, S, \ldots, B)$ with $\rho(A_{17}, T) = m_1, \rho(S, B) = m_2$ and $m_1 + m_2 \leq 16$, or the path from A_{17} to B is $(A_{17}, \ldots, S, T, \ldots, B)$ with $\rho(A_{17}, S) = k_1, \rho(T, B) = k_2$ and $k_1 + k_2 \leq 16$. Then there exists a travel from A to B $(A, \ldots, T, \ldots, A_{18}, A_{17}, \ldots, S, \ldots, B)$ using at most $l_1 + m_1 + 1 + l_2 + m_2 < 34$ flights or $(A, \ldots, T, \ldots, B)$ using at most $l_1 + k_2 < 34$ flights. Contradiction.