Chapter 9

Improper Integrals and Advanced Limit Techniques

In this chapter we will develop some more advanced techniques for computing limits. In the first section, we will look at many determinate forms. In the next section we compute many limits of indeterminate forms $0/0$ or ∞/∞ using a very useful—but not universally applicable—technique known as L'Hôpital's Rule. Next we look at other indeterminate forms, revisiting $\infty - \infty$ and $0 \cdot \infty$, and introducing for the first time $(0^+)^0$, ∞^0, and $(1)^\infty$ using a simple strategy to find related (but not equivalent) fractional forms on which to apply L'Hôpital's Rule, and how to use the conclusions there to determine the original limit. Finally we apply these and previous methods to so-called *improper integrals*, meaning those for which we relax the rules that the integrand must be a continuous function $f(x)$ on a closed and bounded interval $[a, b]$. Examples of useful integrals $\int_a^b f(x)\,dx$ which break these rules include

$$\int_{-2}^{2} \frac{1}{x^{2/3}}\,dx, \qquad \int_0^1 \ln x\,dx, \qquad \int_1^\infty \frac{1}{x^2}\,dx, \qquad \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx,$$

among many others. In all of the above, either the function $f(x)$ is not continuous in the entire range of integration $[a, b]$ (particularly because of the presence of vertical asymptotes there), or those "limits of integration" a and b are not finite. We will eventually make sense of such integrals by intuitive approaches which will require all of our previous limit methods, as well as those developed here in the earlier sections of this Chapter.

9.1 Some Asymptotics of Functions

It will be important to be able to spot the "forms" of the limits we will encounter in this chapter and beyond. This was also the case in Chapter 3, but we have encountered many more functions since then.

In this section we first take note of the behaviors of functions near vertical asymptotes, and as $|x| \to \infty$. These behaviors we will collectively call the *asymptotics* of the functions. In our first subsection we will look at particular functions. Then we will look at compositions and combinations of these functions, in cases where we can somewhat quickly read the behavior of these more complicated functions as following from the behaviors of the underlying functions.

The functions in Figure 9.1, page 671 have all been encountered earlier in the text. Taking these functions in turn, we can list their asymptotic and other limiting behaviors, as read off of their respective graphs.

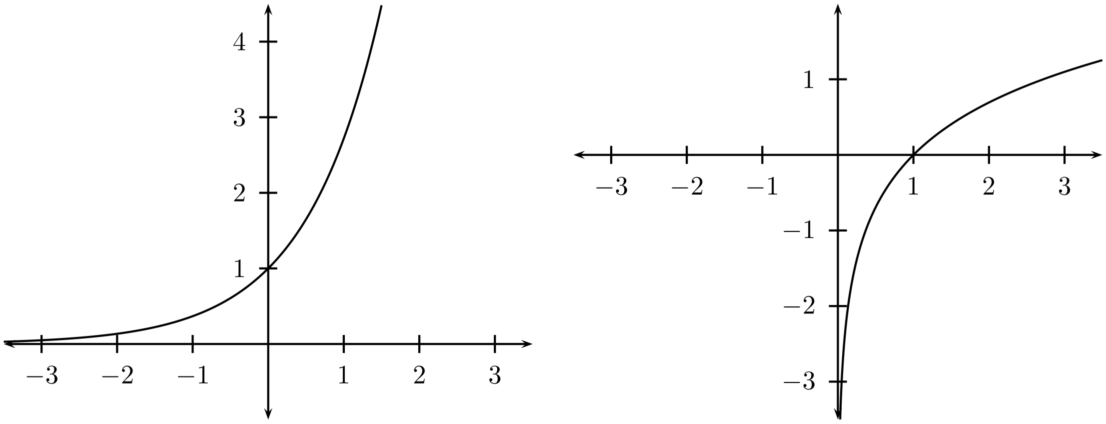

Figure 9.1.a: Graph of $y = e^x$

Figure 9.1.b: Graph of $y = \ln x$

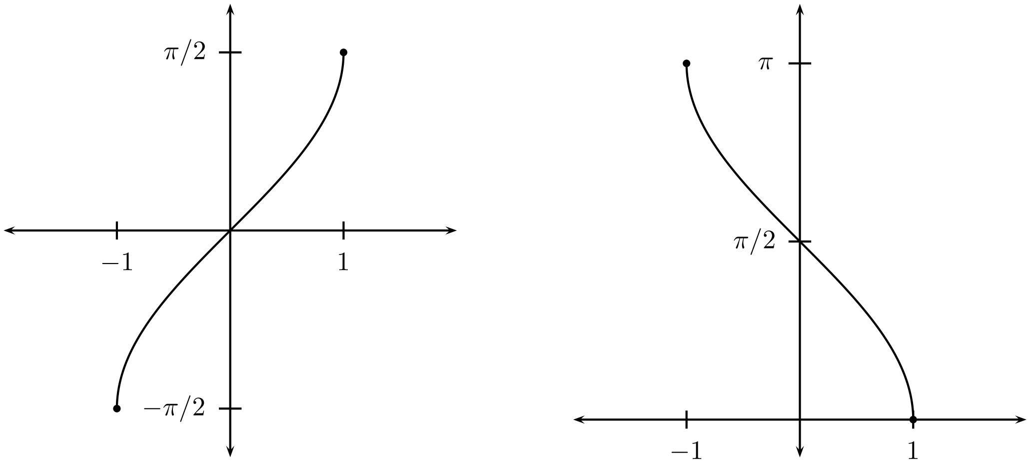

Figure 9.1.c: Graph of $y = \sin^{-1} x$

Figure 9.1.d: Graph of $y = \cos^{-1} x$

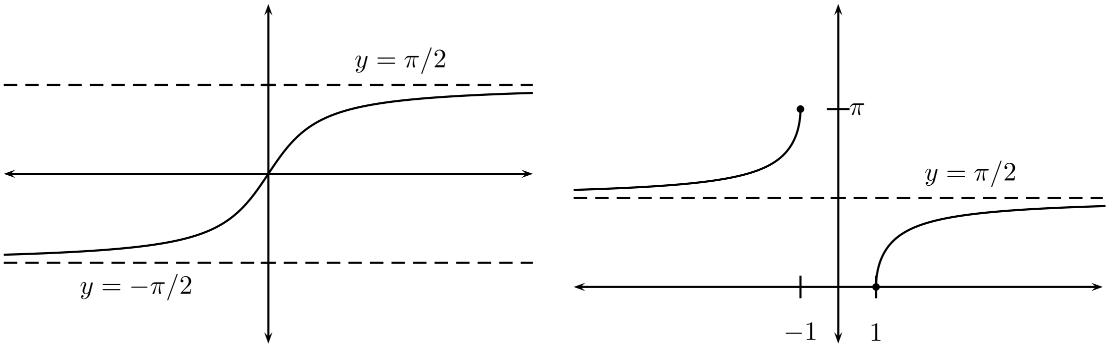

Figure 9.1.e: Graph of $y = \tan^{-1} x$

Figure 9.1.f: Graph of $y = \sec^{-1} x$

Figure 9.1: Graphs of some important functions, illustrating asymptotic behaviors.

1. e^x:
   - (a) $x \to \infty \implies e^x \to \infty$
   - (b) $x \to -\infty \implies e^x \to 0^+$
2. $\ln x$ (note the relationship between this and e^x, which is the inverse of $\ln x$):
   - (a) $x \to \infty \implies \ln x \to \infty$
   - (b) $x \to 0^+ \implies \ln x \to -\infty$
3. $\sin^{-1} x$:
   - (a) $x \to -1^+ \implies \sin^{-1} x \to \left(-\frac{\pi}{2}\right)^+$
   - (b) $x \to 1^- \implies \sin^{-1} x \to \left(\frac{\pi}{2}\right)^-$
4. $\cos^{-1} x$:
   - (a) $x \to -1^+ \implies \cos^{-1} x \to \pi^-$
   - (b) $x \to 1^- \implies \cos^{-1} x \to 0^+$
5. $\tan^{-1} x$:
   - (a) $x \to -\infty \implies \tan^{-1} x \to \left(-\frac{\pi}{2}\right)^+$
   - (b) $x \to \infty \implies \tan^{-1} x \to \left(\frac{\pi}{2}\right)^-$
6. $\sec^{-1} x$:
   - (a) $x \to -\infty \implies \sec^{-1} x \to \left(\frac{\pi}{2}\right)^+$
   - (b) $x \to \infty \implies \sec^{-1} x \to \left(\frac{\pi}{2}\right)^-$

These give rise to limit forms, so "$e^\infty = \infty$," "$e^{-\infty} = 0^+$," and so on. It is also important to remember where these, and all basic functions, are continuous (i.e., for which values a we have $x \to a \implies f(x) \to f(a)$), and what their limiting behavior (from any direction) is at any points within their domains, or approachable from within their domains.

The above limiting behaviors are all clear from the graphs. We can now apply these to more complicated limits where relevant.

Example 9.1.1 *Consider the following limits:*

$$\lim_{x\to\infty} \tan^{-1}(\ln x) \overset{\tan^{-1}\infty}{=\!=\!=\!=} \frac{\pi}{2},$$
$$\lim_{x\to 0^+} \tan^{-1}(\ln x) \overset{\tan^{-1}(-\infty)}{=\!=\!=\!=\!=} -\frac{\pi}{2}.$$

When looking at these combinations of functions, it is important to look "inside-out" to see how the "inner" and "component" functions behave in the limit. To emphasize this, we will sometimes illustrate computations such as the above in ways such as the following:

$$\lim_{x\to\infty} \tan^{-1}(\underbrace{\ln x}_{\substack{\downarrow\\ \infty}}) = \frac{\pi}{2}, \qquad \lim_{x\to 0^+} \tan^{-1}(\underbrace{\ln x}_{\substack{\downarrow\\ -\infty}}) = -\frac{\pi}{2}.$$

Yet another way to compute these limits is through substitution. For instance, in this last limit we can write set $u = \ln x$, so that $x \to 0^+ \implies u \to -\infty$, (similar to Theorem 3.9.4, page 275) and so

$$\lim_{x\to 0^+} \tan^{-1}(\underbrace{\ln x}_{u}) = \lim_{u\to-\infty} \tan^{-1} u = -\frac{\pi}{2}.$$

It does help to have the graphs of both of these functions (ln and $\tan^{-1}$) in mind when performing the computations. However it is more important to know the limiting behaviors, as they may not appear immediately.

Example 9.1.2 *Consider the following limits:*

- $\lim\limits_{x\to\infty} \ln x = \infty.$
- $\lim\limits_{x\to\infty} \ln(\ln x) \overset{\ln(\infty)}{=\!=\!=\!=} \infty.$

- $\lim\limits_{x\to\infty}\ln(\ln(\ln x)) \overset{\ln(\ln(\infty)))}{\underset{i.e.,\ \ln(\infty)}{=\!=\!=\!=}} \infty$

With this last one, we can also argue "pictorially." As $x\to\infty$*, we have*

$$\begin{tikzcd} \infty & \\ \ln(\underbrace{\ln(\overbrace{\ln x})}) \arrow[u] \arrow[d] \arrow[r, "\therefore"] & \infty \\ \infty & \end{tikzcd}$$

The function $\ln(\ln(\ln x))$ *does indeed grow to infinity as* x *does so, but the growth is very, very slow. If we wish this function to be greater than 10, we would need* $x > e^{e^{e^{10}}}$*, which is a real number to be sure, but well beyond most readily available computational devices. (Note that* $e^{10}\approx 22026$*, and* $e^{e^{10}}\approx 10^{9565}$*, so the exponential of that number is astronomically huge.)*[1]

Note that neither $\lim_{x\to-\infty}\tan^{-1}(\ln x)$ *nor* $\lim_{x\to 0^-}\tan^{-1}(\ln x)$ *exist, since the natural logarithm is not defined as* x *"travels the path" prescribed by the limit.*

The above visual representation lends itself better to handwritten mathematics, for instance in notebooks or on a chalkboard, so we will make somewhat limited use of it here.

Many of the limits which occur naturally in subsequent sections are compositions of functions as above. We will consider a few more of these before moving on to limiting behaviors of other combinations (products, quotients, etc.) of functions.

Example 9.1.3 *Consider the following limit computions. In many, it is important to keep in mind the graphs of the functions involved.*

- $\lim\limits_{x\to\frac{\pi}{2}^-}\ln(\cos x) \overset{\ln 0^+}{=\!=\!=\!=} -\infty.$
- $\lim\limits_{x\to\infty}\cos\left(\dfrac{x^2-9x+15}{3x^2+6x-1100}\right) \overset{\cos\frac{1}{3}}{=\!=\!=\!=} \cos\dfrac{1}{3}.$

  Note: cosine is continuous everywhere, including at $\frac{1}{3}$*, so if its argument approaches* $\frac{1}{3}$*, then the cosine approaches* $\cos\frac{1}{3}\approx 0.9449$ *(with the angle 1/3 measured in radians).*
- $\lim\limits_{x\to\infty}\dfrac{1}{\tan^{-1}x-\frac{\pi}{2}} \overset{\frac{1}{0^-}}{=\!=\!=\!=} -\infty.$

  Note $x\to\infty \implies \tan^{-1}x\to(\pi/2)^- \implies \tan^{-1}x-\frac{\pi}{2}\to 0^-$.
- $\lim\limits_{x\to\infty}e^{e^x} \overset{e^\infty}{=\!=\!=\!=} \infty.$
- $\lim\limits_{x\to-\infty}e^{e^x} \overset{e^{0^+}}{=\!=\!=\!=} 1.$
- $\lim\limits_{x\to1^+}\ln(\sec^{-1}x) \overset{\ln(0^+)}{=\!=\!=\!=} -\infty.$
- $\lim\limits_{x\to0}\ln(\sin x) \overset{\ln(0^\pm)}{=\!=\!=\!=}$ *Does Not Exist.*
- $\lim\limits_{x\to\infty}\sqrt{\ln x} \overset{\sqrt{\infty}}{=\!=\!=\!=} \infty.$
- $\lim\limits_{x\to\infty}\dfrac{1}{\ln x} \overset{1/\infty}{=\!=\!=\!=} 0.$
- $\lim\limits_{x\to0^+}\dfrac{1}{\ln x} \overset{1/(-\infty)}{=\!=\!=\!=} 0.$
- $\lim\limits_{x\to\frac{\pi}{2}^-}e^{\tan x} \overset{e^\infty}{=\!=\!=\!=} \infty.$

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[1] In fact, numbers that large are routinely found in statistical mechanics, also known as modern thermodynamics, where it is not so important how large is a particular number *per se,* but what is often crucial is how it compares to another (large) number.

- $\displaystyle\lim_{x\to\frac{\pi}{2}^+} e^{\tan x} \overset{e^{-\infty}}{=\!=\!=} 0.$
- $\displaystyle\lim_{x\to 0^+} \sec^{-1}\ln x \overset{\sec^{-1}(-\infty)}{=\!=\!=\!=\!=} \frac{\pi}{2}.$
- $\displaystyle\lim_{x\to 1^+} \ln(\ln x) \overset{\ln(0^+)}{=\!=\!=\!=} -\infty.$
- $\displaystyle\lim_{x\to\infty} \sin x \overset{\sin\infty}{=\!=\!=}$ *Does Not Exist.*
- $\displaystyle\lim_{x\to\infty} \sec^{-1}\ln x \overset{\sec^{-1}\infty}{=\!=\!=\!=\!=} \frac{\pi}{2}.$
- $\displaystyle\lim_{x\to\infty} (x+\sin x) \overset{\infty+B}{=\!=\!=} \infty.$

The last two limits above require some explanation. That $\lim_{x\to\infty}\sin x$ does not exist is because the sine function continues to oscillate between -1 and 1 as $x\to\infty$, and thus never approaches settling on a particular value. The last limit is infinite because the sine, while oscillating, nevertheless is *bounded* in its output within $[-1,1]$, and can thus never overcome the effects of the term x, and we know $x\to\infty$ in the limit.

Learning how to compute these types of limits is a matter of remembering how the individual functions are behaving for the limiting value of the variable (x in all the above), and how they are "put together" to form the complete function. The arguments are not difficult, but do require practice.

Exercises

Compute the following limits.

1. $\displaystyle\lim_{x\to\infty} 2^x$
2. $\displaystyle\lim_{x\to\infty} 2^{-x}$
3. $\displaystyle\lim_{x\to\infty} \left(\frac{2}{3}\right)^x$
4. $\displaystyle\lim_{x\to\infty} \left(\frac{2}{3}\right)^{-x}$
5. $\displaystyle\lim_{x\to\infty} 1.001^x$
6. $\displaystyle\lim_{x\to\infty} .99^x$
7. $\displaystyle\lim_{x\to\infty} .99^{1-x}$
8. $\displaystyle\lim_{x\to\infty} \tan^{-1}\sqrt{x}$
9. $\displaystyle\lim_{x\to\infty} \tan^{-1}(1-e^x)$
10. $\displaystyle\lim_{x\to 1^+} \ln(\sec^{-1}x)$
11. $\displaystyle\lim_{x\to\infty} \frac{\cos x}{x}$
12. $\displaystyle\lim_{x\to\infty} \tan^{-1}(e^x)$
13. $\displaystyle\lim_{x\to\infty} \tan^{-1}\left(e^{-x}\right)$
14. $\displaystyle\lim_{x\to\infty} e^{-x^2}$
15. $\displaystyle\lim_{x\to-\infty} e^{-x^2}$
16. $\displaystyle\lim_{x\to\infty} \ln\left(\frac{x^2+9}{x^2-1}\right)$
17. $\displaystyle\lim_{x\to\infty} \ln\frac{1}{x}$

9.2 L'Hôpital's Rule

In this section we introduce a very powerful technique for computing limits of the forms $0/0$ and ∞/∞, and their variants.[2] These are very important limits in calculus for several reasons, but overall they measure the relative shrinkage or growth of two quantities in the limit: the numerator and the denominator of the function. In particular, the derivative (where it exists) is a $0/0$ form limit:

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}.$$

It measures the instantaneous change in $f(x)$ (as $x \to a$) per unit change in x. Clearly knowing that the limit is of form $0/0$ is not enough to tell us the actual value of the limit, or nearly all of our derivatives would be the same (which they are not). In the earlier development of derivatives, we used algebraic and, in the case of the sine function, sandwich theorem and geometric arguments to compute such limits. With l'Hôpital's Rule we will have another tool available.

The forms $0/0$ were, in fact, cases where the outcome depended upon which function approached zero faster: if the numerator was much faster in approaching zero, then its effect was stronger and the fraction shrunk in size to zero; if the denominator was much faster in approaching zero, then its effect was stronger and the fraction blew up to produce a limit of ∞, $-\infty$, or perhaps nonexistent if both "blowups" were present; and if the effects were proportional, some finite limit could be the result. So for instance we had limits like

$$\begin{aligned}
&\lim_{x\to 0} \frac{\sin x}{x} = 1,\\
\implies &\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x} = 1 \cdot \frac{0}{2} = 0,\\
\implies &\lim_{x\to 0} \frac{x^2}{(1-\cos x)^2} = \lim_{x\to 0} \frac{1}{\left(\frac{1-\cos x}{x}\right)^2} \overset{1/0^+}{=\!=\!=} \infty.
\end{aligned}$$

The first was proven using a geometric argument, the second followed from the first with some algebra (namely multiplying both the numerator and the denominator of the second limit's function by $(1+\cos x)$), and the third from the second by noting its form is $1/0^+$. So in the first, the tending towards zero in the numerator and denominator is at very nearly the same rate (for small x), in the second the numerator approaches zero much faster than the denominator, and in the third it is the denominator which approaches zero faster.

A similar ratio-style comparison of rates occurs with limits of the form ∞/∞. The following are fairly routine:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x}{x^2+1} &= 0,\\
\lim_{x\to\infty} \frac{x^2}{x+1} &= \infty,\\
\lim_{x\to\infty} \frac{3x^2+9}{5x^2-x+1} &= \frac{3}{5}.
\end{aligned}$$

The trick to seeing these was to factor powers of x from the numerator and denominator and see which effects can cancel each other in the tug-of-war between the numerator and denominator, and analyze the behavior of the terms that are left. So for example one can write

$$\lim_{x\to\infty} \frac{3x^2+9}{5x^2-x+1} \overset{\infty/\infty}{=\!=\!=\!=} \lim_{x\to\infty} \frac{x^2\left(3+\frac{9}{x^2}\right)}{x^2\left(5-\frac{1}{x}+\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{3+\frac{9}{x^2}}{5-\frac{1}{x}+\frac{1}{x^2}} = \frac{3+0}{5-0+0} = \frac{3}{5}.$$

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[2] By variants we mean that, for instance, $0^+/0^-$, $\infty/(-\infty)$, etc.

At times, however, these algebraic analyses are either unwieldy or at least appear to fail, and our main tool to break through such an impass is l'Hôpital's Rule.[3] The beauty of l'Hôpital's Rule is that it measures the relative rates of convergence of the numerator and denominator towards their limiting values by comparing their derivatives, which after all do measure their rates of change. So the rule is intuitive, though not trivial to verify. The rule is stated as follows:

Theorem 9.2.1 *Suppose that for some limiting value of x we have $f(x), g(x) \to 0$ or $f(x), g(x) \to \infty$. Then, if there exists $L \in \mathbb{R} \cup \{-\infty, \infty\}$ such that $f'(x)/g'(x) \to L$ exists, it follows that $f(x)/g(x) \to L$ as well.*

So for instance, if we have a $0/0$ form limit we can look instead at the limits of the derivatives of the numerator and denominator of our original function. Similarly with ∞/∞. Rephrased:

1. If $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)}$ is of $0/0$ form, then (see notes below)

$$\lim_{x\to a} \frac{f(x)}{g(x)} \overset{0/0}{=\!=} \lim_{x\to a} \frac{f'(x)}{g'(x)} \qquad \textbf{if this second limit exists.}$$

2. If $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)}$ is of ∞/∞ form, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} \overset{\infty/\infty}{=\!=\!=} \lim_{x\to a} \frac{f'(x)}{g'(x)} \qquad \textbf{if this second limit exists.}$$

Of course the hope is that the second limit will not only exist (or the theorem does not apply!), but will be easier to compute. Several observations are in order before demonstrating the theorem.

1. It is important that the second limit exists. If not, some other method has to be found to compute the original limit, which might or might not exist. In other words, if the second limit does not exist, that is not enough to say anything about the original limit.
2. **This rule requires $0/0$ or ∞/∞ form, or the rule has no input on the limit.** (Attempting to use the rule for other limit forms is a very common mistake.)
3. If a previous method seems to show potential for success, it should also be considered.
4. The above also works for one-sided limits, limits at infinity, and when the second limit is infinite.

This last point shows how robust l'Hôpital's Rule is, in that it can work for many cases. However, there are cases for which it does not, and it is certainly not a rule to use on every limit.

Now we look at our first example of how one usually employs the rule.

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[3] Guillaume Franois Antoine, Marquis de l'Hôpital (originally l'Hospital, 1661–February 2, 1704), French mathematician, who apparently never claimed actual credit for the rule he published anonymously, which many believe was in fact discovered by his teacher, the Swiss mathematician Johann Bernoulli (July 27, 1667–January 1, 1748), who was also in his employ. Apparently the two agreed that Bernoulli would produce mathematical results and l'Hôpital would publish them after paying Bernoulli. L'Hôpital gave some nonspecific credit to Bernoulli for contributing to the work, in particular for the 1696 textbook *Analysis of the Infinitely Small to Understand Curves (l'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes)*, so it is not enitrely clear who discovered the rule. However, l'Hôpital was apparently a respectable mathematician in his own right, solving the famous "Brachistochrone Problem," for finding the "curve of fastest descent," independently (and fairly simultaneously) from Sir Isaac Newton and others.

Example 9.2.1 $\lim_{x\to 0}\frac{\sin x}{x}\overset{0/0}{\underset{\text{LHR}}{=\!=\!=}}\lim_{x\to 0}\frac{\left(\frac{d\,\sin x}{dx}\right)}{\left(\frac{dx}{dx}\right)}=\lim_{x\to 0}\frac{\cos x}{1}=\frac{\cos 0}{1}=1.$ *(See footnote.[4])*

Our application of the rule does not read in the same logical order that the logic of the rule is stated. In fact when we invoke the rule, as in our first "=," we are doing so provisionally; we write "=" until we are proven wrong (if ever) by the subsequent limit not existing. Because the new limit did exist our first "=" is vindicated, and we declare the computation finished (see the original statement of l'Hôpital's Rule). We will see how to deal with cases where the second limit does not exist, or is no easier to compute, later in this section.

Other examples where we had previous techniques also illustrate the validity of l'Hôpital's Rule, such as the next two examples.

Example 9.2.2 $\lim_{x\to 2}\frac{x^2-4}{x^2+x-6}\overset{0/0}{\underset{\text{LHR}}{=\!=\!=}}\lim_{x\to 2}\frac{2x}{2x+1}=\frac{4}{5}.$

Alternatively, $\lim_{x\to 2}\frac{x^2-4}{x^2+x-6}\overset{0/0}{\underset{\text{ALG}}{=\!=\!=}}\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(x+3)}\overset{0/0}{\underset{\text{ALG}}{=\!=\!=}}\lim_{x\to 2}\frac{x+2}{x+3}=\frac{4}{5}.$

Example 9.2.3 $\lim_{x\to\infty}\frac{2x+5}{3x+4}\overset{\infty/\infty}{\underset{\text{LHR}}{=\!=\!=}}\lim_{x\to\infty}\frac{2}{3}=\frac{2}{3}.$

Alternatively, $\lim_{x\to\infty}\frac{2x+5}{3x+4}\overset{\infty/\infty}{\underset{\text{ALG}}{=\!=\!=}}\lim_{x\to\infty}\frac{x\left(2+\frac{5}{x}\right)}{x\left(3+\frac{4}{x}\right)}\overset{\infty/\infty}{\underset{\text{ALG}}{=\!=\!=}}\lim_{x\to\infty}\frac{2+\frac{5}{x}}{3+\frac{4}{x}}=\frac{2+0}{3+0}=\frac{2}{3}.$

In fact any of the limits at infinity for rational functions can be computed either way, though (as we discuss for other examples later), we may need to employ l'Hôpital's Rule several times for a rational function with higher-degree polynomials in the numerator and denominator if we wish to use that rule. In fact neither method is as efficient as simply learning the asymptotic rules for rational functions, namely comparing degrees of the numerator and denominator, but those rules were based upon the algebraic approach, and l'Hôpital's Rule is usually unnecessarily lengthy.

The more obvious utility of l'Hôpital's Rule is in cases where our algebraic attempts fail to cancel the factors which cause the common limiting behavior of the numerator and denominator. We compute two such limits in the next example. Note that $\ln x$ can not algebraically "cancel," in full or in part, with a power of x.

Example 9.2.4 *Consider the following limits. (See Figure 9.2.)*

- $\lim_{x\to\infty}\frac{\ln x}{x}\overset{\infty/\infty}{\underset{\text{LHR}}{=\!=\!=}}\lim_{x\to\infty}\frac{\frac{1}{x}}{1}\overset{0/1}{=\!=\!=}0.$
- $\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}\overset{\infty/\infty}{\underset{\text{LHR}}{=\!=\!=}}\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}\overset{0/0}{\underset{\text{ALG}}{=\!=\!=}}\lim_{x\to\infty}\left(\frac{1}{x}\cdot 2\sqrt{x}\right)=\lim_{x\to\infty}\frac{2}{\sqrt{x}}\overset{2/\infty}{=\!=\!=}0.$

The above example illustrates that $\ln x$ grows towards ∞ much more slowly than x (as $x\to\infty$), and in fact much more slowly than $\sqrt{x}$. While the first limit seems very reasonable in light of Figure 9.2, because $y=\ln x$ and $y=x$ seem to diverge so quickly and irreversibly,

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[4] In fact there is a circular argument here, because we used geometry to prove that $\lim_{x\to 0}\frac{\sin x}{x}=1$ (Theorem 3.9.2, page 270), and used that limit to prove that $\frac{d}{dx}\sin x=\cos x$ (Theorem 4.2.6, page 318), so we can hardly use that derivative to prove the original limit! So Example 9.2.1 is not a proof of the limit, yet we consider this example merely to demonstrate l'Hôpital's Rule and to verify that it is consistent with our original result for this limit.

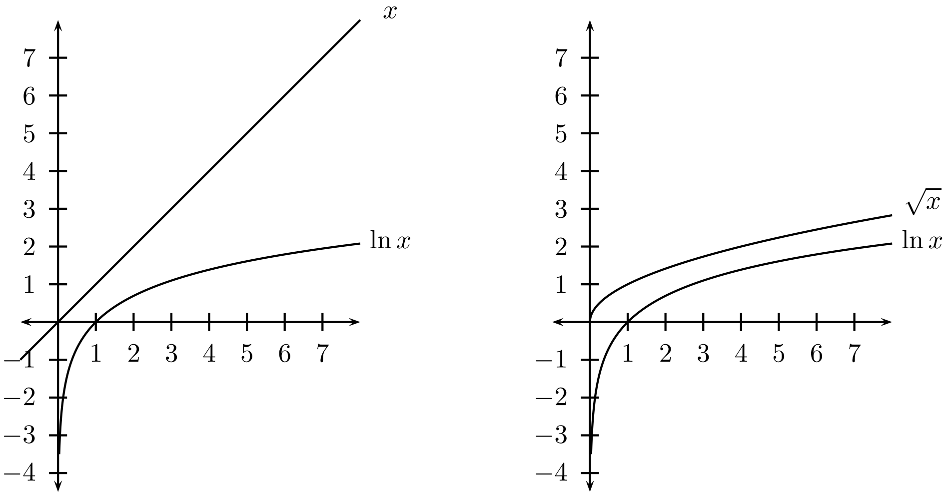

Figure 9.2: Graphs for Example 9.2.4, showing relative growths of $y = \ln x$ versus $y = x$, and then $y = \ln x$ versus $y = \sqrt{x}$. In fact $y = \ln x$ grows more slowly than any positive power of x, i.e., x^s for any $s > 0$, a fact that can be proven using l'Hôpital's Rule. The graph, at least for the scale used here, is not adequately convincing that $(\ln x)/\sqrt{x} \to 0$ as $x \to \infty$, so we need a tool like l'Hôpital's Rule (or a very different scale for our graph!).

by contrast the shapes of $y = \ln x$ and $y = \sqrt{x}$ appear too similar to draw the same conclusion from their graphs, especially with the scale used in the figure. However, when we compare their slopes, respectively $\frac{1}{x}$ and $\frac{1}{2\sqrt{x}}$, we see the latter shrinks much more slowly than the former, as our limit computations verify. l'Hôpital's Rule gives us a more sensitive tool to compare the numerator and denominator, at least for the second problem in Example 9.2.4 above.

In fact $\ln x$ grows more slowly than any positive power of x, i.e., more slowly than x^s for any $s > 0$. The proof is left to the exercises. Note that we can also "flip" the function in that previous limit and still use l'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{\sqrt{x}}{\ln x}\overset{\infty/\infty}{\underset{\text{LHR}}{=\!=}}\lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{x}}\overset{0/0}{\underset{\text{ALG}}{=\!=}}\lim_{x\to\infty}\left(\frac{1}{2\sqrt{x}}\cdot x\right)=\lim_{x\to\infty}\frac{\sqrt{x}}{2}\overset{\infty/2}{=\!=}\infty.$$

Since the final limit actually does exist—albeit as ∞—we conclude the original limit is also ∞. Again this shows $\sqrt{x}$ grows much faster than $\ln x$, as $x \to \infty$.

How quickly $\ln u \to 0$ as as $u \to 1$ is also interesting, though can be more complicated.

Example 9.2.5 *(See Figure 9.3.)* $\displaystyle\lim_{x\to 0}\frac{\ln(1+7x)}{x}\overset{0/0}{\underset{\text{LHR}}{=\!=}}\lim_{x\to 0}\frac{\frac{1}{1+7x}\cdot 7}{1}=\frac{7}{1}=7.$

Though not vital for computing such limits, some intuition regarding l'Hôpital's Rule for 0/0 forms can be seen from this and similar examples. Indeed, for any 0/0-form example where both functions have legitimate linear approximations at the limit point, such as the functions $\ln(1+7x)$ and x as $x \to 0$, these approximations help illustrate the nature of l'Hôpital's Rule for such 0/0 cases.[5] For our two relvant functions above, we have $\ln(1+7x) \approx 0+7x = 7x$, i.e., $l(x) = 0+7x$ is the linear approximation of $\ln(1+7x)$ near $x = 0$, and $y = x$ is its own linear approximation (everywhere). From these it seems we can state that $\ln(1+7x)/x \approx 7x/x = 7$, this being arguably

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[5] Recall $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ is the *linear approximation* of $f(x)$ centered at x_0, assuming $f'(x_0)$ exists. If $f(x_0) = 0$, then the linear approximation is just $f(x) \approx f'(x_0)(x - x_0)$.

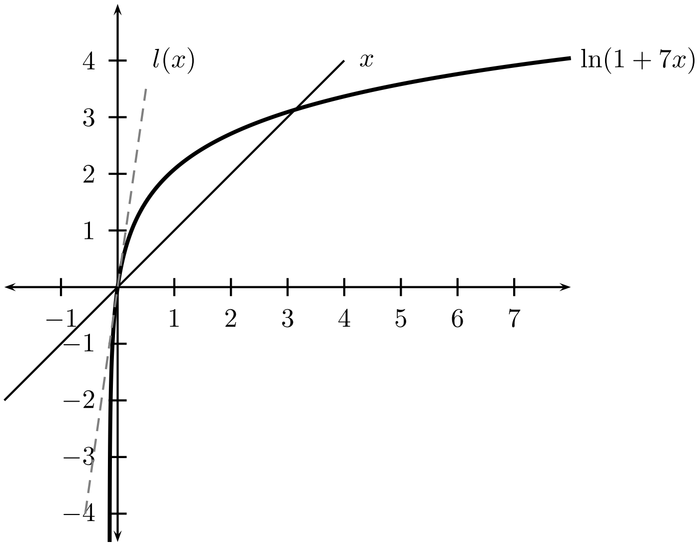

Figure 9.3: Graphs for Example 9.2.5, showing relative behaviors of $y = \ln(7x+1)$ versus $y = x$, particularly as $x \to 0$. The slope of the former is 7 at $x = 0$, where the latter has slope 1. At $x = 0$, the linear approximation of $\ln(1+7x)$ is given by the dashed line $l(x) = 0 + 7x$, while the linear approximation for $y = x$ is itself. From those it is a bit more believable that $(\ln(1+7x))/x \to 7$ as $x \to 0$, since for $x \to 0$ we have $(\ln(1+7x))/x \approx l(x)/x = 7x/x = 7$, and in the limit the original ratio gets ever closer to this approximation, yielding a final limit of 7, coinciding with the ratio of derivatives of $\ln(1+7x)$ and x at $x = 0$.

a better and better approximation as $x \to 0$, giving some intuition why l'Hôpital's Rule makes sense. However we also need the derivatives to approach $\frac{d}{dx}\ln(1+7x)\big|_{x=0}$ and $\frac{dx}{dx}\big|_{x=0}$ as $x \to 0$, for this approximation to equal the new limit (of the ratio of derivatives) at the limit point, but for this example both derivatives are continuous at $x = 0$ (the interested reader should verify) so the limit of derviatves computed in the example and described in l'Hôpital's Rule is consistent with this observation, at least for this case. See Figure 9.3.

If the limit point is not $x = 0$, then the same kind of ratio of derivatives occurs again, which a glance at the linear approximations will show (because the "$f(x_0)$" term is zero, see Footnote 5, page 678). If the limit point is not a point of continuity of the functions (and their derivatives), or the limit point is not finite, then this intuition needs to be modified. We will not look at all possible cases of l'Hôpital's Rule here, as it would be a rather long distraction.

There are cases in which l'Hôpital's Rule, in whatever form, is invoked multiple times to compute a single limit. If the forms continue to be $0/0$ or ∞/∞ until the final limit, and it exists, then it will equal the original. It is important to verify that we have one of these forms at each step we apply l'Hôpital's Rule.

Example 9.2.6 *Consider the following limits.*

- $\displaystyle\lim_{x\to 0}\frac{e^x-x-1}{x^2}\overset{0/0}{\underset{\text{LHR}}{=}}\lim_{x\to 0}\frac{e^x-1}{2x}\overset{0/0}{\underset{\text{LHR}}{=}}\lim_{x\to 0}\frac{e^x}{2}\overset{e^0/2}{\underset{\text{LHR}}{=}}=\frac{1}{2},$
- $\displaystyle\lim_{x\to 0}\frac{\sin x-xe^{x^2}}{x^2}\overset{0/0}{\underset{\text{LHR}}{=}}\lim_{x\to 0}\frac{\cos x-(x\cdot 2xe^{x^2}+e^{x^2})}{2x}=\lim_{x\to 0}\frac{\cos x-2x^2e^{x^2}-e^{x^2}}{2x}$

$$\overset{0/0}{\underset{\text{LHR}}{=}}\lim_{x\to 0}\frac{-\sin x-(2x^2\cdot 2xe^{x^2}+4xe^{x^2})-2xe^{x^2}}{2}=\frac{0}{2}=0.$$

There are times when l'Hôpital's Rule seems useless because either the limit of derivatives does not exist, or it is no simpler than the original limit. In such cases, usually previous methods need to be employed.

Example 9.2.7 *If we try to compute* $\displaystyle\lim_{x\to\infty}\frac{2^x}{3^x}$*, it is quite simple using earlier methods:* $\displaystyle\lim_{x\to\infty}\frac{2^x}{3^x}=\lim_{x\to\infty}\left(\frac{2}{3}\right)^x=0$*, since we know that* $y=a^x\to 0^+$ *as* $x\to\infty$ *if* $a\in(0,1)$*.*

But if we attempt to use l'Hôpital's Rule to compute this limit, we have

$$\lim_{x\to\infty}\frac{2^x}{3^x}\overset{\infty/\infty}{\underset{\text{LHR}}{=}}\lim_{x\to\infty}\frac{2^x\ln 2}{3^x\ln 3}\overset{\infty/\infty}{\underset{\text{LHR}}{=}}\lim_{x\to\infty}\frac{2^x(\ln 2)^2}{3^x(\ln 3)^2}=\cdots,$$

which, while true, does not get us any closer to the correct answer.

Example 9.2.8 *Attempting to compute* $\displaystyle\lim_{x\to\infty}\frac{\sqrt{3x^2+4}}{x}$ *using l'Hôpital's Rule finds us no better off than we were at the start.*

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+4}}{x}\overset{\infty/\infty}{\underset{\text{LHR}}{=}}\lim_{x\to\infty}\frac{\frac{6x}{2\sqrt{3x^2+4}}}{1}=\lim_{x\to\infty}\frac{3x}{\sqrt{3x^2+4}}.$$

Clearly this is no simpler than the original. We could attempt a strategy where we try to "solve for the limit" algebraically, as we sometimes did for the antiderivative with integration by parts—here we would have $L=\frac{3}{L}$*—except we are not sure (without previous methods) that we are dealing with a finite, or even existent, limit* L *for which to solve. So instead we look to previous methods (note* $x>0$*):*

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+4}}{x}\overset{\infty/\infty}{\underset{\text{ALG}}{=}}\lim_{x\to\infty}\frac{|x|\sqrt{3+\frac{4}{x^2}}}{x}=\lim_{x\to\infty}\sqrt{3+\frac{4}{x^2}}=\sqrt{3+0}=\sqrt{3}.$$

In fact, that is consistent with our algebraic equation $L=3/L\implies L^2=3$*. Note that there is another strategy, where we use the continuity of the square root to move the "lim" inside:* $\displaystyle\lim_{x\to\infty}\frac{\sqrt{3x^2+4}}{x}\overset{\infty/\infty}{\underset{\text{ALG}}{=}}=\lim_{x\to\infty}\sqrt{\frac{3x^2+4}{x^2}}=\sqrt{\lim_{x\to\infty}\frac{3x^2+4}{x^2}}=\sqrt{3}$*. As with l'Hôpital's Rule, bringing the limit inside the radical required that the final limit existed, which it did.*

Next we look at an example where the new limit (of derivatives) in l'Hôpital's Rule does not exist, though the original limit does.

Example 9.2.9 *Consider* $\displaystyle\lim_{x\to\infty}\frac{\sin x+x}{\cos x+x}$*. First note that* $-1\le\sin x\le 1$*, and* $-1\le\cos x\le 1$*, so as* $x\to\infty$ *we have* $\sin x+x\to\infty$ *and* $\cos x+x\to\infty$*. This can be seen by using the sandwich theorem, but we really only need one side of it (the other being* $1+x\ge\sin x+x$*) because we have*

$$\sin x+x\ge -1+x\to\infty\qquad\text{as } x\to\infty,$$

implying the greater quantity $\sin x + x \to \infty$ *as well. Similarly for* $\cos x + x$.

Using our old methods, where we let "B" stand for any bounded term when useful (such as for $\sin x$ *since* $|\sin x| \le 1$*), we might write*

$$\lim_{x\to\infty} \frac{\sin x + x}{\cos x + x} \xlongequal[\text{ALG}]{\infty/\infty} \lim_{x\to\infty} \frac{x\left(\frac{\sin x}{x} + 1\right)}{x\left(\frac{\cos x}{x} + 1\right)} = \lim_{x\to\infty} \frac{\frac{\sin x}{x} + 1}{\frac{\cos x}{x} + 1} \xlongequal{\frac{(B/\infty)+1}{(B/\infty)+1}} \frac{0+1}{0+1} = 1.$$

However, we should note that if we attempted l'Hôpital's Rule, we would be next analyzing $\displaystyle\lim_{x\to\infty} \frac{\cos x + 1}{-\sin x + 1}$*, which does not exist because both numerator and denominator are oscillating, but not in a way in which these effects would cancel. This is readily seen by the periodic nature of the new quotient, but we have another reason to conclude this second limit does not exist: the fact that* $-\sin x + 1 \to 0^{\pm}$ *for many instances where* $\cos x + 1 \to 1$ *(namely, when* $x \to \frac{\pi}{2} + 2n\pi$, $n \in \{0, \pm 1, \pm 2, \cdots\}$*), making the function periodically undefined as* x *increases, again leading us to conclude the new limit does not exist (though the original does).*

The above example demonstrates that we can not use l'Hôpital's Rule directly if the second limit does not exist. However, there are times we can still use l'Hôpital's Rule for some cases, if we are willing to look at both left and right limits.

Example 9.2.10 *Consider* $\displaystyle\lim_{x\to 0} \frac{\sin x}{x^2}$*. If we first attempt to use l'Hôpital's Rule, we would write* $\displaystyle\lim_{x\to 0} \frac{\sin x}{x^2} \xlongequal[\text{LHR?}]{0/0} \lim_{x\to 0} \frac{\cos x}{2x}$ *does not exist* $(1/0^{\pm})$*, so l'Hôpital's Rule does not, technically, apply. However, we note that if we had taken left and right limits separately, l'Hôpital's Rule would apply to each:*

$$\lim_{x\to 0^-} \frac{\sin x}{x^2} \xlongequal[\text{LHR}]{0/0} \lim_{x\to 0^-} \frac{\cos x}{2x} \xlongequal{1/0^-} -\infty,$$

$$\lim_{x\to 0^+} \frac{\sin x}{x^2} \xlongequal[\text{LHR}]{0/0} \lim_{x\to 0^+} \frac{\cos x}{2x} \xlongequal{1/0^+} \infty.$$

From these computations we can see that left and right limits for the original function differ (one is $-\infty$ *and the other* ∞*), so we conclude that* $\displaystyle\lim_{x\to 0} \frac{\sin x}{x^2}$ *does not exist.*

L'Hôpital's Rule is a very useful method to add to one's arsenal, but it does require discretion for when to use it, and when not to use it. Sometimes it is useful, or just necessary, for only part or parts of the limit, but not the whole limit collectively.

Example 9.2.11 *Compute* $\displaystyle\lim_{x\to 0} \frac{(x^2 + 7x - 9)\sin x}{4x^5 + 6x^2 + 7x}$.

<u>Solution</u>*: Here we factor the function in question into the product of two functions:*

$$\lim_{x\to 0} \frac{(x^2 + 7x - 9)\sin x}{4x^5 + 6x^2 + 7x} = \lim_{x\to 0} \left[\frac{x^2 + 7x - 9}{4x^4 + 6x + 7} \cdot \frac{\sin x}{x}\right] = \left(\lim_{x\to 0} \frac{x^2 + 7x - 9}{4x^4 + 6x + 7}\right) \underbrace{\left(\lim_{x\to 0} \frac{\sin x}{x}\right)}_{0/0,\ \text{LHR}}$$

$$= \frac{-9}{7} \cdot \lim_{x\to 0} \frac{\cos x}{1} = -\frac{9}{7} \cdot 1 = -\frac{9}{7}.$$

Note that this was only allowed because both limits existed, though you can always factor out a any part that has a finite, nonzero limit and find the limit of what remains, existing or not.

In fact we could have used l'Hôpital's Rule on the entire function above, but we would have endured needless complications, in this case the product rule repeatedly.

The following limit is already broken into three summed parts, and while we could combine them into one fraction with which to use l'Hôpital's Rule, it is much easier to analyze the terms separately, and if each limit exists as a finite number, then the limit of the sum will be the sum of the limits.

Example 9.2.12 *Compute* $\displaystyle\lim_{x\to\infty}\left(\tan^{-1}x+\frac{(\ln x)^2}{x}-e^{1/x}\right)$.

Solution: Here again we break the function into pieces that have known limits, and a piece that needs to be analyzed further.

$$\begin{aligned}\lim_{x\to\infty}\left(\tan^{-1}x+\frac{(\ln x)^2}{x}-e^{1/x}\right)&=\frac{\pi}{2}+\underbrace{\left(\lim_{x\to\infty}\frac{(\ln x)^2}{x}\right)}_{\infty/\infty,\ \text{LHR}}-e^0=\frac{\pi}{2}+\left(\lim_{x\to\infty}\frac{(2\ln x)\cdot\frac{1}{x}}{1}\right)-1\\&=\frac{\pi}{2}-1+\underbrace{\lim_{x\to\infty}\frac{2\ln x}{x}}_{\infty/\infty,\ \text{LHR}}=\frac{\pi}{2}-1+\lim_{x\to\infty}\frac{\frac{2}{x}}{1}=\frac{\pi}{2}-1+\frac{0}{1}\\&=\frac{\pi}{2}-1.\end{aligned}$$

We do have to be a little careful in breaking a limit's function into various parts. For instance, if doing so gives us another indeterminate form we have to adopt another strategy, often recombining the function in some expedient way (as in the next section). For instance, if the "pieces" of the function have limits that would collective read $\infty-\infty$, or $0\cdot\infty$, etc., then we have to find some way to combine them to either cancel competing influences (as much as possible), or employ other techniques, including but not limited to l'Hôpital's Rule.

It also happens sometimes that a $0/0$ or ∞/∞ form occurs within a function, and we are wise to attempt to use LHR "inside" the function to eventually lead us to the final conclusion regarding a limit's value.

Example 9.2.13 *Compute* $\displaystyle\lim_{x\to\infty}\cos\left(\frac{\ln x}{x}\right)$.

Solution: Here we do not have a function which is a quotient per se, but we can compute the limit inside the function, and if it approaches a point for which we can conclude to have a determinate "form," we can deduce the actual limit. Here,

$$\begin{aligned}&\lim_{x\to\infty}\frac{\ln x}{x}\overset{0/0}{\underset{\text{LHR}}{=\!=\!=}}\lim_{x\to\infty}\frac{\frac{1}{x}}{1}\overset{(1/\infty)/1}{=\!=\!=\!=}0\\\Longrightarrow\ &\lim_{x\to\infty}\cos\left(\frac{\ln x}{x}\right)\overset{\cos 0}{=\!=\!=}1.\end{aligned}$$

At this point a warning regarding l'Hôpital's Rule is in order. While the rule involves derivatives and quotients, it does not invoke the quotient rule; we are not taking the derivative of the function itself, but of its numerator and denominator separately. Misreading l'Hôpital's Rule and applying the quotient rule is another common mistake for novice calculus students. However the more common mistake is to try to use LHR when we do not have a $0/0$ or ∞/∞ form.

Note: Historical information was obtained from the internet site "Wikipedia," located at the URL `http://www.wikipedia.com`.

Exercises

Compute the following limits.

1. Compute $\lim\limits_{x\to 0}\dfrac{\tan^{-1}x}{x}$.
2. Compute $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\tan x-1}{x-\frac{\pi}{4}}$.
3. Compute $\lim\limits_{x\to 1}\dfrac{\tan^{-1}x-\frac{\pi}{4}}{x-1}$.
4. Compute $\lim\limits_{x\to 1}\dfrac{\ln x}{x-1}$.
5. Compute $\lim\limits_{x\to 0^+}\dfrac{\ln x}{\left(\frac{1}{x}\right)}$.
6. Compute $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\sin 3x}$.
7. Compute $\lim\limits_{x\to 0}\dfrac{e^x-\frac{1}{1-x}}{x^2}$ two ways:
   (a) By using l'Hôpital's Rule directly.
   (b) By simplifying the fraction and then using l'Hôpital's Rule.
8. Compute $\lim\limits_{x\to 0}\dfrac{\sin x}{xe^{x^2}}$.
9. Show that $\lim\limits_{x\to\infty}\dfrac{\ln x}{x^s}=0$ for any $s>0$.
10. Show that $\lim\limits_{x\to\infty}\dfrac{e^x}{x^n}=\infty$ for any $n\in\{1,2,3,\cdots\}$. (You will need to "employ" l'Hôpital's Rule n times, or at least declare some pattern eventually.)
11. Compute $\lim\limits_{x\to 0^+}\dfrac{\ln(1+x)}{x}$.
12. Compute $\lim\limits_{x\to\infty}\dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$, two ways.
    (a) With l'Hôpital's Rule directly.
    (b) Using the previous limit. (Let $u=\frac{1}{x}$.)
13. Compute $\lim\limits_{x\to\infty}\dfrac{3x}{\sqrt{4x^2+9x+7}}$.
14. Compute $\lim\limits_{x\to\infty}\dfrac{2^x}{x^2}$.
15. Compute $\lim\limits_{x\to 0}\dfrac{2^x}{x^2}$.
16. Compute $\lim\limits_{x\to 0}\dfrac{x^2}{2^x}$.
17. Compute $\lim\limits_{x\to\infty}xe^{-x}$.
18. Compute $\lim\limits_{x\to 0}\tan^{-1}\left(\dfrac{\sin x}{x}\right)$.
19. Compute $\lim\limits_{x\to\infty}\sin\left(\dfrac{\pi x^2-4x+6}{1-2x^2}\right)$.

9.3 Other Indeterminate Forms

Besides the forms $0/0$ and ∞/∞, for which we can use old methods or l'Hôpital's Rule, there are other forms which are also indeterminate. Here we will be mostly interested in the following, which the reader should confirm seem indeterminate (meaning just knowing that the limit is of such a form is not enough to determine the limit's value).

1. $\infty - \infty$.
2. $0 \cdot \infty$.
3. $(0^+)^0$.
4. ∞^0.
5. 1^∞.

While the methods of Chapter 3 will sometimes work here, we will usually need something that attacks these new problems with slightly different strategies. So far our only new tool has been l'Hôpital's Rule, and we will use it again here extensively, but it requires a quotient. So to summarize our method of attack with these problems, we attempt to rewrite the problems so that we have a quotient which can be analyzed by l'Hôpital's Rule or previous strategies. In fact, having a quotient is often useful even if we do not need l'Hôpital's Rule.

9.3.1 $\infty - \infty$, $0 \cdot \infty$

We also include variants such as $0 \cdot (-\infty)$, $0^+ \cdot (-\infty)$, etc., and when the order of the terms is switched (e.g., $\infty \cdot 0$). For obvious reasons, do not include definite forms like $\infty + \infty = \infty$, or $\infty \cdot \infty = \infty$, or for that matter $0 - 0 = 0$, $0 \cdot 0 = 0$.

Example 9.3.1 *Compute* $\displaystyle\lim_{x\to\infty}\left[\sqrt{x^2-5x+100}-x\right]$.

Solution: Here we have $\infty-\infty$, *since* $x^2-5x+100\to\infty$ *and therefore* $\sqrt{x^2-5x+100}\to\infty$. *In fact this type of limit was already analyzed in Chapter 3 (specifically Example 3.8.8, page 260). Note that we do solve this by re-writing the limit as a quotient.*

$$\begin{aligned}
\lim_{x\to\infty}\left[\sqrt{x^2-5x+100}-x\right] &\overset{\infty-\infty}{=\!=\!=} \lim_{x\to\infty}\left[\left(\sqrt{x^2-5x+100}-x\right)\cdot\frac{\sqrt{x^2-5x+100}+x}{\sqrt{x^2-5x+100}+x}\right]\\
&= \lim_{x\to\infty}\frac{x^2-5x+100-x^2}{\sqrt{x^2-5x+100}+x} = \lim_{x\to\infty}\frac{-5x+100}{\sqrt{x^2-5x+100}}\\
&= \lim_{x\to\infty}\frac{x\cdot\left(-5-\frac{100}{x}\right)}{x\cdot\left(\sqrt{1-\frac{5}{x}+\frac{100}{x^2}}+1\right)} = \lim_{x\to\infty}\frac{-5-\frac{100}{x}}{\sqrt{1-\frac{5}{x}+\frac{100}{x^2}}+1}\\
&= \frac{-5-0}{\sqrt{1-0+0}+1} = -\frac{5}{2}.
\end{aligned}$$

The example above did not require l'Hôpital's Rule, and in fact l'Hôpital's Rule is not helpful even after we have a quotient. (See Example 9.2.8, page 680.)

The next example is of the form $-\infty \cdot 0^+$ (sometimes just written $-\infty \cdot 0$, which is similar to $\infty \cdot 0$ except for the sign).

Example 9.3.2 *Compute* $\lim\limits_{x\to 0^+} x\ln x$.

Solution: Since this is of form $0\cdot(-\infty)$, *which is indeterminate, our strategy is to first algebraically rewrite the function as a quotient. This will give us a form on which we can apply l'Hôpital's Rule.*

$$\lim_{x\to 0^+} x\ln x \overset{0\cdot(-\infty)}{\underset{\text{ALG}}{=\!=\!=}} \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{-\infty/\infty}{\underset{\text{LHR}}{=\!=\!=}} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}\left[\frac{1}{x}\cdot\frac{-x^2}{1}\right] = \lim_{x\to 0^+}(-x) = 0.$$

In the above example, the influence of $x\to 0$ is stronger than that of $\ln x\to-\infty$. To reach this conclusion, we needed a quotient to compare properly the growth of these two functions, because a quotient allows tools such as l'Hôpital's Rule.

It should be pointed out that we could have placed the natural logarithm function in the denominator, but that would not give us a satisfactory application of l'Hôpital's Rule:

$$\lim_{x\to 0^+} x\ln x \overset{0\cdot(-\infty)}{\underset{\text{ALG}}{=\!=\!=}} \lim_{x\to 0^+} \frac{x}{(\ln x)^{-1}} \overset{0^+/0^-}{\underset{\text{LHR}}{=\!=\!=}} \lim_{x\to 0^+} \frac{1}{-(\ln x)^{-2}\cdot\frac{1}{x}} = \lim_{x\to 0^+} \frac{-(\ln x)^2}{x}.$$

We see that our new limit is no simpler, and is arguably harder than the the original.

Example 9.3.3 *Compute* $\lim\limits_{x\to\infty} x^2e^{-x}$.

Solution: This is actually simpler than the previous example.

$$\lim_{x\to\infty} x^2e^{-x} \overset{\infty\cdot 0}{\underset{\text{ALG}}{=\!=\!=}} \lim_{x\to\infty} \frac{x^2}{e^x} \overset{\infty/\infty}{\underset{\text{LHR}}{=\!=\!=}} \lim_{x\to\infty} \frac{2x}{e^x} \overset{\infty/\infty}{\underset{\text{LHR}}{=\!=\!=}} \lim_{x\to\infty} \frac{2}{e^x} \overset{2/\infty}{=\!=\!=} 0.$$

Example 9.3.4 *Consider the following* $\infty\cdot 0$ *limits. Note that sometimes we have no choice but to put a logarithmic term in the new denominator:*

- $$\begin{aligned}&\lim_{x\to\infty} x\ln(\cos(1/x)) \overset{\infty\cdot 0}{\underset{\text{ALG}}{=\!=\!=}} \lim_{x\to\infty} \frac{\ln(\cos(1/x))}{1/x} \overset{0/0}{\underset{\text{LHR}}{=\!=\!=}} \lim_{x\to\infty} \frac{\frac{1}{\cos(1/x)}\cdot(-\sin(1/x))\cdot\frac{-1}{x^2}}{\frac{-1}{x^2}}\\ &\quad= \lim_{x\to\infty}\tan(1/x) = \tan 0 = 0.\end{aligned}$$

- $$\begin{aligned}&\lim_{x\to\infty} (\ln x)\ln(\cos 1/x) \overset{\infty\cdot 0}{\underset{\text{ALG}}{=\!=\!=}} \lim_{x\to\infty} \frac{\ln(\cos 1/x)}{(\ln x)^{-1}} \overset{0/0}{\underset{\text{LHR}}{=\!=\!=}} \lim_{x\to\infty} \frac{\frac{1}{\cos 1/x}\cdot(-\sin 1/x)\frac{-1}{x^2}}{\frac{-1}{(\ln x)^2}\cdot\frac{-1}{x}}\\ &\quad= \lim_{x\to\infty} \frac{-\tan(1/x)(\ln x)^2}{x} \overset{0\cdot\infty/\infty}{\underset{\text{ALG}}{=\!=\!=}} \lim_{x\to\infty} \frac{-(\ln x)^2}{x\cot(1/x)}\\ &\quad\overset{\infty/\infty}{\underset{\text{LHR}}{=\!=\!=}} \lim_{x\to\infty} \frac{-2\ln x\cdot\frac{1}{x}}{x\left(-\csc(1/x)\cdot\frac{-1}{x^2}\right)+\cot(1/x)}\cdot\frac{x}{x} = \lim_{x\to\infty} \frac{-2\ln x}{\csc(1/x)+x\cot(1/x)}\\ &\quad\overset{-\infty/\infty}{\underset{\text{LHR}}{=\!=\!=}} \lim_{x\to\infty} \frac{\frac{-2}{x}}{-\csc(1/x)\cot(1/x)\cdot\frac{-1}{x^2}+\cot(1/x)+x\left(-\csc^2(1/x)\cdot\frac{-1}{x^2}\right)}\cdot\frac{x^2}{x^2}\\ &\quad= \lim_{x\to\infty} \frac{-2x}{\csc(1/x)\cot(1/x)+x^2\cot(1/x)+x\csc^2(1/x)}\\ &\quad\overset{-\infty/\infty}{\underset{\text{LHR}}{=\!=\!=}} \lim_{x\to\infty}\left[-2\Bigg/\left(-\csc(1/x)\cot(1/x)\cdot\frac{-1}{x^2}\cot(1/x)\right.\right.\\ &\qquad -\csc(1/x)\cdot\left(-\csc^2(1/x)\cdot\frac{-1}{x^2}\right)+2x\cot(1/x)+x^2\left(-\csc^2(1/x)\cdot\frac{-1}{x^2}\right)+\csc^2(1/x)\\ &\qquad \left.\left.+x\cdot 2\csc(1/x)\left(-\csc(1/x)\cot(1/x)\cdot\frac{-1}{x^2}\right)\right)\right] \overset{-2/\infty}{=\!=\!=} 0.\end{aligned}$$

(It was important that all terms in the denominator have are added, that the trigonometric functions are all approaching ∞*, and so is* $1/x^2$*.) Note how we do what we can to get* $\ln x$ *by itself, without other factors, so that it does not reappear when we invoke l'Hôpital's Rule.*

Note also that since $x \to \infty$ *faster than* $\ln x \to \infty$*, we could have used the first limit computation above to anticipate this one. In other words, if* $x\ln(\cos(1/x)) \to 0$*, and for large enough* x *we have* $0 \le \ln x < x$*, we expect* $(\ln x)(\ln(\cos(1/x))) \to 0$ *as well.*

Finally note that we can not use l'Hôpital's Rule for the $0 \cdot \infty/\infty$*, because we need to know we have* $0/0$ *or* ∞/∞*. After a bit of algebra, we got* ∞/∞*, used l'Hôpital's Rule, then used more algebra until we had a determinate form* $0/\infty = 0$*. Most problems are not that involved.*

- $$\lim_{x\to\infty} x^2\ln(\cos(1/x)) \overset{\infty\cdot 0}{\underset{\text{ALG}}{=\!=\!=}} \lim_{x\to\infty}\frac{\ln(\cos(1/x))}{x^{-2}} \overset{0/0}{\underset{\text{LHR}}{=\!=\!=}} \lim_{x\to\infty}\frac{\frac{1}{\cos(1/x)}\cdot(-\sin(1/x))\cdot\frac{-1}{x^2}}{-2x^{-3}}$$
  $$= \lim_{x\to\infty}\frac{\tan(1/x)}{-2\cdot\frac{1}{x}} \overset{0/0}{\underset{\text{LHR}}{=\!=\!=}} \lim_{x\to\infty}\frac{\sec^2(1/x)\cdot\frac{-1}{x^2}}{-2\cdot\frac{-1}{x^2}} = \lim_{x\to\infty}\frac{1}{-2\cos^2(1/x)} = \frac{1}{-2}.$$

- $$\lim_{x\to\infty} x^3\ln(\cos(1/x)) = \lim_{x\to\infty} x\cdot x^2\ln(\cos(1/x)) \overset{\infty\cdot(-1/2)}{=\!=\!=\!=} -\infty.$$

  We could have performed this last limit just like the previous two limits, but we can also use the information from its immediate predecessor to make shorter work of its computation. If instead we had computed it with the same method as before, after algebra and two invocations of l'Hôpital's Rule, we would have had

  $$\lim_{x\to\infty} x^3\ln(\cos(1/x)) = \cdots = \lim_{x\to\infty}\frac{\sec^2(1/x)\cdot\frac{-1}{x^2}}{6x^{-3}} = \lim_{x\to\infty}\frac{-x}{6\cos^2(1/x)} \overset{-\infty/(6\cdot 1)}{=\!=\!=\!=} -\infty.$$

If this last limit seems surprising, note that $\cos(1/x) < 1$ for large x, and so $\ln(\cos(1/x)) < \ln 1 = 0$, so the form is more precisely stated as $\infty \cdot 0^-$.

The above limits from Example 9.3.2 to this last one illustrate again that $0 \cdot \infty$ form is indeterminate (more information than just the form is required), and that the general strategy is to rewrite the function as a ratio (fraction), though there is some cleverness as to the form of the ratio that will help us to find the actual limit. The algebraic rewriting may be necessary in intermediate steps as well. And of course we can only use l'Hôpital's Rule if we have a variation of $0/0$ or ∞/∞ forms.

9.3.2 1^∞, 0^0, ∞^0 and similar forms.

The technique for all of these is to apply the natural logarithm to the function, thus bringing the exponent down as a factor. Some adjustment must be made, because we are then calculating the limit of the natural logarithm of the original function, but that is relatively easy (but crucial) to deal with.

Example 9.3.5 *Compute* $\displaystyle\lim_{x\to\infty}\left(1+\frac{7}{x}\right)^x$.

<u>Solution</u>: This is of the form 1^∞*. If we define* y *to be the function inside the limit, then*

$$\lim_{x\to\infty}\underbrace{\left(1+\frac{7}{x}\right)^x}_{y} = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y}.$$

Once we know that $\ln y \to L$, we would have $y \to e^L$, assuming L is finite. (Even if L is infinite we will have ways of dealing with this.)

$$\begin{aligned}
&\textit{Set} \quad y = \left(1+\frac{7}{x}\right)^x \\
&\implies \ln y = \ln\left(1+\frac{7}{x}\right)^x \\
&\implies \ln y = x\ln\left(1+\frac{7}{x}\right).
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to\infty}\ln y &= \lim_{x\to\infty} x\ln\left(1+\frac{7}{x}\right) \overset{\infty\cdot 0}{=\!=\!=} \lim_{x\to\infty}\frac{\ln\left(1+\frac{7}{x}\right)}{\frac{1}{x}} \underset{\text{LHR}}{\overset{0/0}{=\!=\!=}} \lim_{x\to\infty}\frac{\frac{1}{1+\frac{7}{x}}\cdot\frac{-7}{x^2}}{\frac{-1}{x^2}} \\
&= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{7}{x}}\cdot\frac{-7}{x^2}}{\frac{-1}{x^2}}\cdot\frac{x^2}{x^2} = \lim_{x\to\infty}\frac{7}{1+\frac{7}{x}} = \frac{7}{1+0} = 7.
\end{aligned}$$

This is not our final answer. So far we know where the natural log of our function approaches: $\ln y \to 7$. The actual function's limit is computed next:

$$\lim_{x\to\infty}\left(1+\frac{7}{x}\right)^x = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} \overset{e^7}{=\!=} e^7.$$

The general strategy is to bring the exponent down as a multiplier, which can be put (in the form of its reciprocal) into a denominator, at which point we can use l'Hôpital's Rule and other techniques. However we must realize that the process has us computing the limit of $\ln y$, if y is our original function, so we must report instead the limit of $y = e^{\ln y}$.

It is worth noting that the "7" in Example 9.3.5 can be replaced by any number. In fact we have the following:

$$\lim_{x\to\infty}\left(1+\frac{\xi}{x}\right)^x = e^{\xi}, \tag{9.1}$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^1 = e. \tag{9.2}$$

These give alternative definitions of e^{ξ} and e, respectively.

Example 9.3.6 *Compute* $\lim\limits_{x\to\infty} x^{1/\sqrt{x}}$.

__Solution__: This is of the form ∞^0 (or more precisely ∞^{0^+}). We compute this limit as in the previous example.

$$\begin{aligned}
&\textit{Set} \qquad y = x^{1/\sqrt{x}} \\
&\implies \qquad \ln y = \ln x^{1/\sqrt{x}} = \frac{\ln x}{\sqrt{x}} \\
&\implies \lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} \underset{\text{LHR}}{\overset{\infty/\infty}{=\!=\!=}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}\cdot\frac{2x}{2x} = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0.
\end{aligned}$$

Thus $\lim\limits_{x\to\infty} x^{1/\sqrt{x}} = \lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} e^{\ln y} = e^0 = 1.$

Example 9.3.7 *Compute* $\lim_{x\to 0^+}(\cos x)^{1/x}$, $\lim_{x\to 0^+}(\cos x)^{1/x^2}$ *and* $\lim_{x\to 0^+}(\cos x)^{1/x^3}$.

<u>Solution</u> *These three are very similar, but we write all the compuations for the sake of comarison. All are of the form* 1^∞.

- $\underline{\lim_{x\to 0^+}(\cos x)^{1/x}:}$

$$\begin{aligned} &\textit{Set} && y=(\cos x)^{1/x}\\ &\Longrightarrow && \ln y=\frac{1}{x}\ln(\cos x)\\ &\Longrightarrow && \lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{\ln(\cos x)}{x}\overset{0/0}{\underset{\text{LHR}}{=\!=}}\lim_{x\to 0^+}\frac{\frac{1}{\cos x}(-\sin x)}{1}=\lim_{x\to 0^+}\frac{-\sin x}{\cos x}=\frac{0}{1}=0.\end{aligned}$$

Thus $\lim_{x\to 0^+}(\cos x)^{1/x}=\lim_{x\to 0^+}y=\lim_{x\to 0^+}e^{\ln y}=e^0=1.$

- $\underline{\lim_{x\to 0^+}(\cos x)^{1/x^2}:}$

$$\begin{aligned} &\textit{Set} && y=(\cos x)^{1/x^2}\\ &\Longrightarrow && \ln y=\frac{1}{x^2}\ln(\cos x)\\ &\Longrightarrow && \lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{\ln(\cos x)}{x^2}\overset{0/0}{\underset{\text{LHR}}{=\!=}}\lim_{x\to 0^+}\frac{\frac{1}{\cos x}(-\sin x)}{2x}=\lim_{x\to 0^+}\frac{-\tan x}{2x}\\ &&& \overset{0/0}{\underset{\text{LHR}}{=\!=}}\lim_{x\to 0^+}\frac{-\sec^2 x}{2}=\frac{-1}{2}.\end{aligned}$$

Thus $\lim_{x\to 0^+}(\cos x)^{1/x^2}=\lim_{x\to 0^+}y=\lim_{x\to 0^+}e^{\ln y}=e^{-1/2}=\frac{1}{\sqrt{e}}.$

- $\underline{\lim_{x\to 0^+}(\cos x)^{1/x^3}:}$

$$\begin{aligned} &\textit{Set} && y=(\cos x)^{1/x^3}\\ &\Longrightarrow && \ln y=\frac{1}{x^3}\ln(\cos x)\\ &\Longrightarrow && \lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{\ln(\cos x)}{x^3}\overset{0/0}{\underset{\text{LHR}}{=\!=}}\lim_{x\to 0^+}\frac{\frac{1}{\cos x}(-\sin x)}{3x^2}=\lim_{x\to 0^+}\frac{-\tan x}{3x^2}\\ &&& \overset{0/0}{\underset{\text{LHR}}{=\!=}}\lim_{x\to 0^+}\frac{-\sec^2 x}{6x}\overset{-1/0^+}{=\!=}-\infty.\end{aligned}$$

Thus $\lim_{x\to 0^+}(\cos x)^{1/x^3}=\lim_{x\to 0^+}y=\lim_{x\to 0^+}e^{\ln y}\overset{e^{-\infty}}{=\!=}0.$

Summarizing, though these were all 1^∞*, we have a case where the influence of the* 1 *is stronger, one where they are relatively equal, and one where the* ∞ *is stronger (coupled with the fact that "1" is in each of these cases precisely stated as* 1^-*):*

$$\begin{aligned}\lim_{x\to 0^+}(\cos x)^{1/x}&=1,\\ \lim_{x\to 0^+}(\cos x)^{1/x^2}&=1/\sqrt{e},\\ \lim_{x\to 0^+}(\cos x)^{1/x^3}&=0.\end{aligned}$$

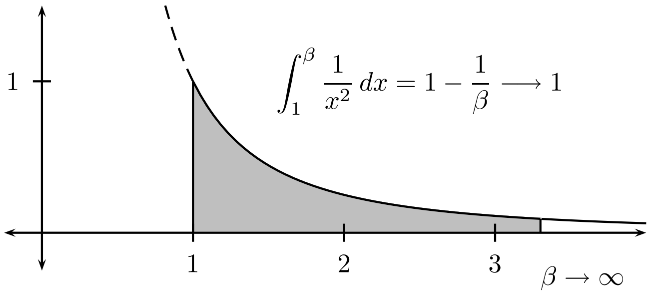

Figure 9.4: Illustration for the method of analyzing $\int_1^\infty \frac{1}{x^2}\,dx$, namely by computing $\int_1^\beta \frac{1}{x^2}\,dx$ for finite β, and letting $\beta \to \infty$.

9.4 Improper Integrals

In this section we look at definite integrals where one or more boundaries of integration—here sometimes called endpoints of integration[6]—is *improper*, meaning not a point of continuity of the integrand. We also consider cases where there is some discontinuity, for instance a vertical asymptote, inside the interval of integration. For any such internal point or endpoint, the basic idea is to "sneak up" on that point, seeing what value an approximating integral approaches if we replace that point by a variable endpoint, which then approaches the offending point using a limit process.

9.4.1 First Examples

Example 9.4.1 *Compute* $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$.

Solution: This integral is "improper" at $x = \infty$, because we can not claim $1/x^2$ is continuous on $[1,\infty]$, since ∞ is not a proper **included** *endpoint for such an interval. Instead we consider $\int_1^\beta \frac{1}{x^2}\,dx$, for $\beta \in (1,\infty)$, so that $1/x^2$ is continuous on $[1,\beta]$ and the Fundamental Theorem of Calculus applies to $\int_1^\beta \frac{1}{x^2}\,dx$, and finally we see the limiting behavior of this definite integral as $\beta \to \infty$:*

$$\lim_{\beta\to\infty}\int_1^\beta \frac{1}{x^2}\,dx = \lim_{\beta\to\infty} \left.\frac{-1}{x}\right|_1^\beta = \lim_{\beta\to\infty}\left[\frac{-1}{\beta} - \frac{-1}{1}\right] = 0 + 1 = 1.$$

From this we conclude that

$$\int_1^\infty \frac{1}{x^2}\,dx = 1.$$

The strategy is illustrated in Figure 9.4. Because the limit of "proper" integrals exists, we express its meaning the following equivalent ways:

---

[6] The usual term is "limit" of integration, which is somewhat unfortunate because it can erroneously conjure an association with limits in the sense of Chapter 3 ("Limits and Continuity"). In fact in this section we will use a lot of limit techniques from that chapter and the previous sections of this chapter, but not always on the original boundaries of our integration, so the phrase "limits of integration" is avoided here.

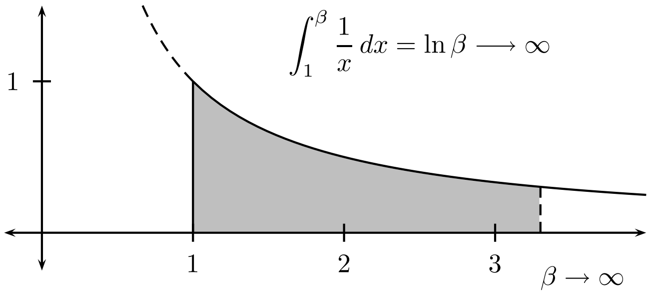

Figure 9.5: Illustration for Example 9.4.2, analyzing $\int_1^\infty \frac{1}{x}\,dx$ by computing the limit of $\int_1^\beta \frac{1}{x}\,dx$ as $\beta \to \infty$. In this case the limit is infinite, so the integral diverges. To be more precise, the integral diverges to infinity.

- $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ *converges to* 1; or, simply
- $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = 1.$

The term *convergence* appears in a few different contexts in this textbook, namely with improper integrals, sequences and series. To converge in our context here means that the relevant limit exists *as a finite number.* To not converge is to *diverge*, meaning either the limit does not exist, or is not finite. More concisely, that an improper integral diverges means that *the limit does not exist as a finite number.*

Note that it is quite possible for the total area to be finite, even though the length of the interval is infinite. With an integral such as in Example 9.4.1 above, what matters is whether or not the function shrinks fast enough as $x \to \infty$ (so the limit for $\beta \to \infty$ is finite).

Example 9.4.2 *Determine if* $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ *converges or diverges, and if it converges compute its value.*

Solution: The technique is the same as the previous example's: we look at closed intervals $[1,\beta]$*, on which the integrand* $1/x$ *is continuous so the Fundamental Theorem of Calculus applies, and we compute the limit of these integrals as* $\beta \to \infty$*:*

$$\lim_{\beta\to\infty}\int_1^\beta \frac{1}{x}\,dx = \lim_{\beta\to\infty} \ln x\Big|_1^\beta = \lim_{\beta\to\infty} \ln\beta \overset{\ln\infty}{=\!=\!=} \infty.$$

Therefore $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ *diverges.*

To be more precise, we would say the above integral *diverges to infinity.* For practical reasons we are mainly interested in integrals that converge (to finite numbers). Moreover, there are those that diverge, but not to ∞ or $-\infty$. All nonconvergent cases like these can be put into one class and labeled as "divergent."

9.4.2 The Background and Method

Recall that the Fundamental Theorem of Calculus tells us, among other things, that if $f(x)$ is continuous on the (finite, closed) interval $[a,b]$ (a and b being real numbers), then

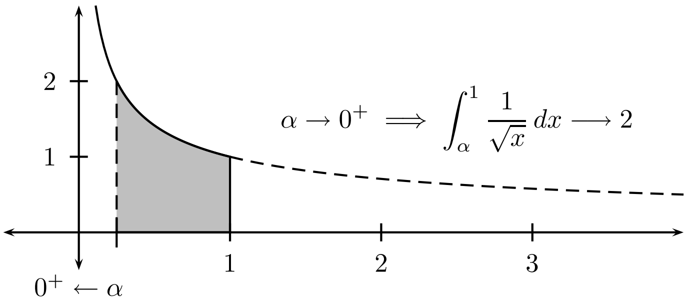

Figure 9.6: Illustration for Example 9.4.3, analyzing $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ by computing the limit of $\int_\alpha^1 \frac{1}{\sqrt{x}}\,dx$ as $\alpha \to 1^+$. In this case the limit is finite, namely 2, so the integral converges to 2.

- there exists a continuous function $F(x)$ defined on $[a,b]$ (and beyond the endpoints if so desired), such that for all $x \in (a,b)$ we have $F'(x) = f(x)$, and

$$\lim_{\Delta x \to 0^+} \frac{F(a+\Delta x) - F(a)}{\Delta x} = f(a),$$
$$\lim_{\Delta x \to 0^-} \frac{F(b+\Delta x) - F(b)}{\Delta x} = f(b).$$

- Moreover, $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

(Recall that, technically $\int_a^b f(x)\,dx$ denotes a limit of Riemann Sums over $[a,b]$ with $\max\{\Delta x_i\} \to 0$.) So the Fundamental Theorem of Calculus requires a closed interval of the form $[a,b]$, and a function which is continuous on that closed interval. Two types of problems with this are dealt with here:

1. one or both of the endpoints is not finite, as in our first two examples above, or
2. the function may have a discontinuity in $[a,b]$, at an endpoint or internally (in (a,b)).

In fact, an integral may have several such features to deal with in assigning a value to it. In all cases, the strategy is to break the integral into subintegrals in which at most one enpoint is "improper" (infinite or a discontinuity, the latter usually a vertical asymptote), and then for each such subintegral we look at definite integrals with one variable endpoint approaching the "improper" endpoint of the subintegral, from the correct side, and such that the Fundamental Theorem of Calculus can be applied to the approximating subintegrals.

We now illustrate this with several more such improper integrals. Our next integral is improper because one of the endpoints is improper. See Figure 9.6.

Example 9.4.3 *Compute, if it converges,* $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$.

<u>Solution</u>: Here the integral is improper at $x = 0$, *where the function has a vertical asymptote. We thus look at definite integrals on* $[\alpha, 1]$ *where* $\alpha \in (0,1)$, *and so we can use the Fundamental Theorem of Calculus with this function on these intervals, and let* $\alpha \to 0^+$.

$$\lim_{\alpha \to 0^+} \int_\alpha^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\alpha \to 0^+} 2\sqrt{x}\Big|_\alpha^1 = \lim_{\alpha \to 0^+} \left(2 - 2\sqrt{\alpha}\right) = 2 - 0 = 2$$
$$\implies \int_0^1 \frac{1}{\sqrt{x}}\,dx = 2.$$

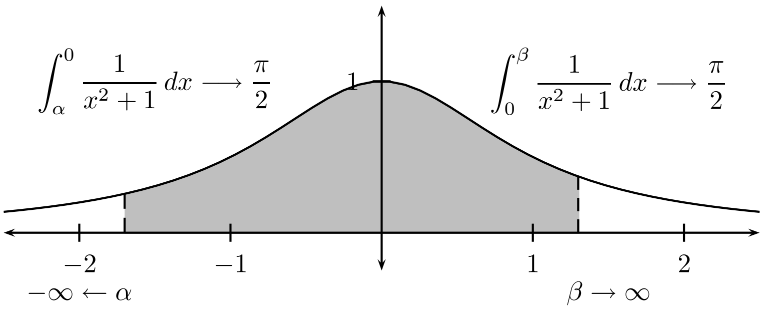

Figure 9.7: Illustration for Example 9.4.4, $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$. This is improper at both $-\infty$ and ∞, so we must consider **separately** $\int_\alpha^0 \frac{1}{x^2+1}\,dx$ and $\int_0^\beta \frac{1}{x^2+1}\,dx$, respectively letting $\alpha \to -\infty$ and $\beta \to \infty$. It turns out both are finite, so the origninal integral is the sum of these.

This last example again shows that, while one dimension of the area may extend without bounds (upwards in this case as $x \to 0^+$), it is possible for the total area to be finite, in this case because of the speed at which the function's graph approaches the vertical asymptote as the function's height increases.

Example 9.4.4 *Compute, if possible,* $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx.$

Solution: The integrand is continuous on all of $\mathbb{R}$, *since* $x^2+1>0$, *but both endpoints are improper. To deal with this we break the range of integration into subintervals in which at most one endpoint is improper. So we look at the following, but taken* **provisionally** *(see Figure 9.7):*

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+1}\,dx.$$

By provisionally we mean that, if both integrals on the right converge, then the equation makes sense and is true. If either of the integrals on the right diverges, then we declare that the original integral diverges. Now we look at these two integrals separately.

$$\begin{aligned}
\int_{-\infty}^{0} \frac{1}{x^2+1}\,dx: &\qquad \lim_{\alpha\to-\infty} \int_\alpha^0 \frac{1}{x^2+1}\,dx = \lim_{\alpha\to-\infty} \tan^{-1}x\Big|_\alpha^0 = \lim_{\alpha\to-\infty}\left(\tan^{-1}0-\tan^{-1}\alpha\right)\\
&\qquad = 0-\frac{-\pi}{2}=\frac{\pi}{2}.\\
\int_{0}^{\infty} \frac{1}{x^2+1}\,dx: &\qquad \lim_{\beta\to\infty} \int_0^\beta \frac{1}{x^2+1}\,dx = \lim_{\beta\to\infty} \tan^{-1}x\Big|_0^\beta = \lim_{\beta\to\infty}\left(\tan^{-1}\beta-\tan^{-1}0\right)\\
&\qquad = \frac{\pi}{2}-0=\frac{\pi}{2}.
\end{aligned}$$

Since both subintegrals converge to finite numbers, the original integral will be their sum:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+1}\,dx = \frac{\pi}{2}+\frac{\pi}{2}=\pi.$$

In this last example we could also make use of the symmetry to get that the two subintegrals are the same, so

$$\int_{-\infty}^{\infty}\frac{1}{x^2+1}\,dx=2\int_0^{\infty}\frac{1}{x^2+1}\,dx=2\cdot\frac{\pi}{2}=\pi.$$

However, the first "=" in the line above is also *provisional*, valid only if the second integral converges.

It should also be pointed out that the endpoint 0 common to both subintegrals was, theoretically, arbitrary. We could have just as easily used $\int_{-\infty}^{1}+\int_1^{\infty}$, or any other real number; the goal is to first break the original integral into two on which there is only one improper endpoint.

Example 9.4.5 *Next we list several examples of how we would have to partition some improper integrals into subintegrals, each having at most one improper endpoint.*

Note that each equation below is **provisional**, *i.e., true if each of the subintegrals converges (to a finite number). In the interests of space and clarity, we will not write the integrands or differentials beyond the initial statement of the integral in question.*

- $\displaystyle\int_{-\infty}^{\infty}\frac{1}{x^2+1}\,dx=\int_{-\infty}^{0}+\int_0^{\infty},$ *Improper at* $-\infty$, ∞.
- $\displaystyle\int_{-1}^{1}\frac{1}{x^{2/3}}\,dx=\int_{-1}^{0}+\int_0^{1},$ *Improper at 0.*
- $\displaystyle\int_{-\infty}^{\infty}\frac{1}{x^3}\,dx=\int_{-\infty}^{-1}+\int_{-1}^{0}+\int_0^{1}+\int_1^{\infty},$ *Improper at* $-\infty$, *0*, ∞.
- $\displaystyle\int_0^{\infty}\frac{1}{x\ln x}\,dx=\int_0^{1/2}+\int_{1/2}^{1}+\int_1^{2}+\int_2^{\infty},$ *Improper at* $0,1,\infty$.
- $\displaystyle\int_0^{4}\frac{1}{(x-1)(x-2)}\,dx=\int_0^{1}+\int_1^{3/2}+\int_{3/2}^{2}+\int_2^{4},$ *Improper at* $1,2$.

9.4.3 Infinite Areas do not Cancel

One has to be careful in making symmetry arguments, especially if claiming that two parts of an integral "cancel." It is not too difficult to see that

$$\int_{-\alpha}^{\alpha}\frac{x}{x^2+1}\,dx=0$$

for any $\alpha\in\mathbb{R}$. While we actually compute this below, it is also easy to visualize for $\alpha>0$, because the integrand is an odd function. See Figure 9.8.[7] However, we can not simply let $\alpha\to\infty$ and conclude that $\int_{-\infty}^{\infty}\frac{x}{x^2+1}\,dx$ will also be zero. The trouble is that the integral on $(-\infty,0]$ diverges to $-\infty$, while the integral on $[0,\infty)$ diverges to ∞, so we can not add them and claim that they cancel. To see why, consider the following computations:

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[7] Recall that f is an *odd* function if and only if $f(-x)=-f(x)$, for all x in the domain of f.

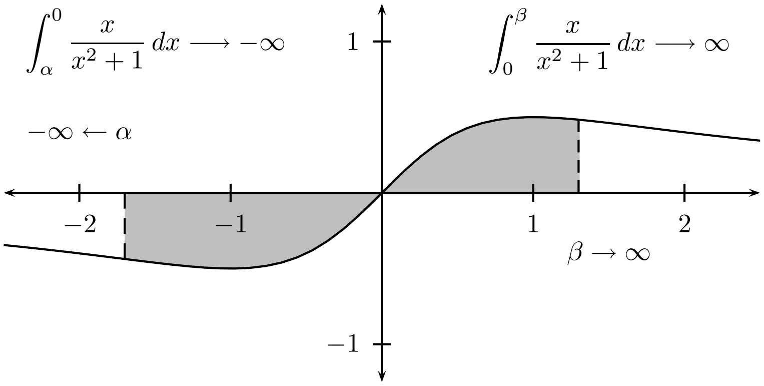

Figure 9.8: Illustration for $\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx$. This is improper at both $-\infty$ and ∞, so we must consider **separately** $\int_\alpha^0 \frac{x}{x^2+1}\,dx$ and $\int_0^\beta \frac{x}{x^2+1}\,dx$, respectively letting $\alpha \to -\infty$ and $\beta \to \infty$. It turns out the first diverges to $-\infty$ and the second to ∞, so the original integral also diverges. It would be incorrect to only look at $\int_{-\alpha}^{\alpha} \frac{x}{x^2+1}\,dx$, and let one limit collectively compute the sum of each, because it would depend heavily on the fact that we would be counting "areas," both positive and negative, precisely the way that they would cancel. However, the areas could be counted other ways collectively to achieve other results—for instance $\int_{-\alpha}^{2\alpha} \frac{x}{x^2+1}\,dx \to \ln 2 \neq 0$ (see text). Though both appear to the naïve observer to "cover" the entire interval $(-\infty,\infty)$ in the limit, they do not return the same "total area." For these and other reasons we decline to assign a value to this improper integral, and instead declare it to be divergent.

$$\lim_{\alpha\to\infty}\int_{-\alpha}^{\alpha}\frac{x}{x^2+1}\,dx = \lim_{\alpha\to\infty}\frac{1}{2}\ln(x^2+1)\bigg|_{-\alpha}^{\alpha} = \lim_{\alpha\to\infty}\left[\frac{1}{2}\ln(\alpha^2+1)-\frac{1}{2}\ln(\alpha^2+1)\right] = 0.$$

$$\lim_{\alpha\to\infty}\int_{-\alpha}^{2\alpha}\frac{x}{x^2+1}\,dx = \lim_{\alpha\to\infty}\frac{1}{2}\ln(x^2+1)\bigg|_{-\alpha}^{2\alpha} = \lim_{\alpha\to\infty}\left[\frac{1}{2}\ln(4\alpha^2+1)-\frac{1}{2}\ln(\alpha^2+1)\right]$$

$$= \lim_{\alpha\to\infty}\frac{1}{2}\ln\frac{4\alpha^2+1}{\alpha^2+1} = \lim_{\alpha\to\infty}\ln\sqrt{\frac{4\alpha^2+1}{\alpha^2+1}} = \ln\sqrt{4} = \ln 2.$$

Note that both integrals cover all of $(-\infty,\infty)$ in the limits, yet they return different values. So the "total area" is different if you count faster on one side of zero than the other. For this and other reasons, we must look at two separate subintegrals for this problem.

First we analyze $\int_0^\infty \frac{x}{x^2+1}\,dx$.

$$\lim_{\beta\to\infty}\int_0^\beta\frac{x}{x^2+1}\,dx = \lim_{\beta\to\infty}\frac{1}{2}\ln(x^2+1)\bigg|_0^\beta = \lim_{\beta\to\infty}\frac{1}{2}[\ln(\beta^2+1)-\ln 1] = \infty.$$

Thus $\int_0^\infty \frac{x}{x^2+1}\,dx$ diverges to ∞. By a similar computation or symmetry, $\int_{-\infty}^0 \frac{x}{x^2+1}\,dx$ diverges to $-\infty$, but that is not necessary to notice this to conclude that $\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$ diverges. Indeed, it was enough that the subintegral $\int_0^\infty \frac{x}{x^2+1}\,dx$ diverges, to conclude the full integral $\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$ does so as well. (The difference is that the full integral does not diverge towards either $-\infty$ or ∞.)

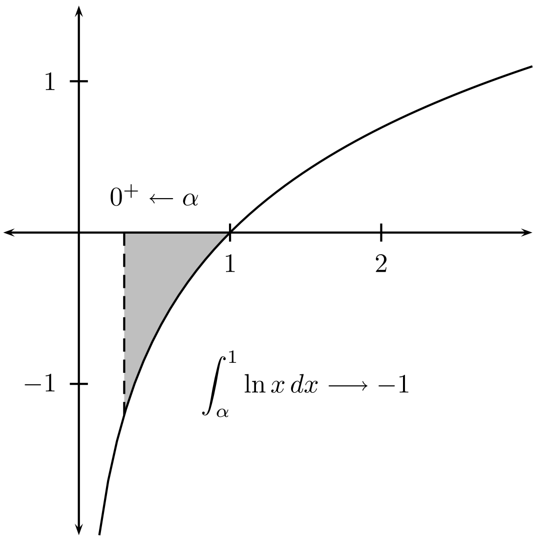

Figure 9.9: Illustration for Example 9.4.6.

9.4.4 More Complicated Integrals

In this subsection we examine ways to organize our solution of an improper integral problem if the antidifferentiation or limit steps are complicated. There are basically five steps to any improper integral computation (some with substeps):

1. Identify improper points within the range of integration, and if necessary break the integral into subintegrals for which only one endpoint is improper.
2. For each of these subintegrals, write a definite integral on a closed interval in which one endpoint is fixed, and the other is allowed to vary, approaching the improper endpoint from within the interval.

> Perform the following two steps collectively, one subintegral at a time, and if any subintegral diverges, we can stop and declare the original integral diverges.

3. Compute the relevant antiderivative for the subintegral.
4. Compute the limit as the variable endpoint approaches the improper endpoint's value from within the interval where the function is continuous (using the Fundamental Theorem of Calculus and limit techniques).
5. If all subintegrals converge, sum them to get the original integral's value.

The antidifferentiation step may be complicated enough that it is advantageous to compute the antiderivative without the endpoints present. Similarly a limit problem may arise which is better not done inline with the general flow of the logic.

Example 9.4.6 *Compute* $\int_0^1 \ln x\,dx$. *(See Figure 9.9.)*

Solution: This is improper at $x = 0$, *so we will eventually look at* $\lim_{\alpha\to 0^+}\int_\alpha^1 \ln x\,dx$, *but first we need to compute the relevant antiderivative.*

$$\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - \int dx = x\ln x - x + C.$$

$$\left.\begin{array}{ll} u = \ln x & dv = dx \\ du = \dfrac{1}{x}\,dx & v = x \end{array}\right|$$

Now we write, provisionally,

$$\int_0^1 \ln x\,dx = \lim_{\alpha\to 0^+}\int_\alpha^1 \ln x\,dx = \lim_{\alpha\to 0^+}(x\ln x - x)\Big|_\alpha^1 = \lim_{\alpha\to 0^+}(0 - 1 - \alpha\ln\alpha - \alpha).$$

The only term in that limit that needs detailed analysis is the $\alpha\ln\alpha$ term, so rather than attempting to wrap it into the other terms (in one unnecessarily complicated fraction), we compute it separately.

$$\lim_{\alpha\to 0^+}\alpha\ln\alpha \overset{0\cdot(-\infty)}{\underset{\text{ALG}}{=\!=\!=}} \lim_{\alpha\to 0^+}\frac{\ln\alpha}{\alpha^{-1}} \overset{-\infty/\infty}{\underset{\text{LHR}}{=\!=\!=}} \lim_{\alpha\to 0^+}\frac{\frac{1}{\alpha}}{-\frac{1}{\alpha^2}} = \lim_{\alpha\to 0^+}(-\alpha) = 0.$$

Filling this result into the limit above it, we get

$$\int_0^1 \ln x\,dx = \lim_{\alpha\to 0^+}\int_\alpha^1 \ln x\,dx = \lim_{\alpha\to 0^+}(0 - 1 - \alpha\ln\alpha - \alpha) = 0 - 1 - 0 - 0 = -1.$$

When we bring techniques from several different topics in calculus to bear on a single problem like that above, it is often useful to work the various dimensions of the problem somewhat separately. So the problem above required that we (1) identify improper points in the range of integration, (2) set up the appropriate limits of integrals, (3) compute the relevant antiderivative(s), (4) compute the limits that emerge from that process, and (5) draw a conclusion from all of that work. Since the antiderivative can be a relatively extensive computation itself, it is better to not attempt it on a definite integral (with endpoints). Computing the desired limits in the same line as the antiderivative computation can also be difficult, especially since there may be several terms in the limits, some of which may need techniques such as l'Hôpital's Rule to compute. One is led to conclude that it is better to break these tasks into sequestered subtasks until their results are needed.

Example 9.4.7 *Compute $\displaystyle\int_0^\infty e^{-x}\sin 4x\,dx$. (See Figure 9.10.)*

<u>*Solution*</u>*: As usual we consider $\displaystyle\int_0^\beta e^{-x}\sin 4x\,dx$, and let $\beta\to\infty$. However it is nontrivial to compute the needed antiderivative by hand, so we make that a "problem within the problem." Recall that this is the type of integral which we integrate by parts twice, and then have an*

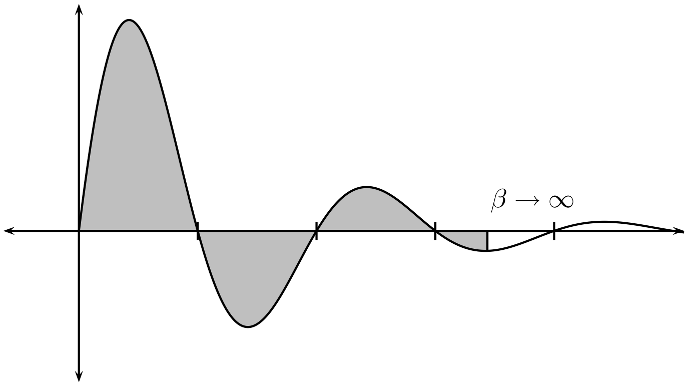

Figure 9.10: Illustration for Example 9.4.7

equation involving the integral which we can solve for the integral.

$$(\boldsymbol{\mathcal{I}}) = \int e^{-x}\sin 4x\,dx = uv - \int v\,du = -e^{-x}\sin 4x + 4\int e^{-x}\cos 4x\,dx$$

$$\left.\begin{array}{ll} u = \sin 4x & dv = e^{-x}\,dx \\ du = 4\cos 4x\,dx & v = -e^{-x} \\ \hline \end{array}\right| = -e^{-x}\sin 4x + 4\left[uv - \int v\,du\right]$$

$$\left.\begin{array}{ll} u = \cos 4x & dv = e^{-x}\,dx \\ du = -4\sin 4x\,dx & v = -e^{-x} \end{array}\right| = -e^{-x}\sin 4x + 4\left[-e^{-x}\cos 4x - 4\int e^{-x}\sin 4x\,dx\right]$$

$$= -e^{-x}\sin 4x - 4e^{-x}\cos 4x - 16(\boldsymbol{\mathcal{I}})$$

$$\Longrightarrow\ 17(\boldsymbol{\mathcal{I}}) = -e^{-x}(\sin 4x + 4\cos 4x) + C_1$$

$$\Longrightarrow\ (\boldsymbol{\mathcal{I}}) = \frac{-e^{-x}}{17}(\sin 4x + 4\cos 4x) + C.$$

With the relevant antiderivative known, we now can now compute

$$\begin{aligned} \lim_{\beta\to\infty}\int_0^\beta e^{-x}\sin 4x\,dx &= \lim_{\beta\to\infty}\left.\frac{-e^{-x}}{17}(\sin 4x + 4\cos 4x)\right|_0^\beta \\ &= \lim_{\beta\to\infty}\left[\frac{-e^{-\beta}}{17}(\sin 4\beta + 4\cos 4\beta + \frac{1}{17}(0+4)\right] \\ &= \lim_{\beta\to\infty}\frac{\sin 4\beta + 4\cos 4\beta}{17e^\beta} + \frac{4}{17} \\ &\overset{(B/\infty)+\frac{4}{17}}{=\!=\!=\!=} 0 + \frac{4}{17} = \frac{4}{17}. \end{aligned}$$

From this we get $\displaystyle\int_0^\infty e^{-x}\sin 4x\,dx = \frac{4}{17}.$

This last example required more difficult antidifferentiation steps. We could also consider integrals with more points where they are improper, but they are more rare in practice. Usually we only need to worry about one or two points where an integral is improper, and once we discover them, we simply break the original into subintegrals with one improper endpoint each, and use a limiting process as developed here.

Examples of applications include certain total work (energy) computations. If we have a force such as gravity, between two stationary objects, varying as the distance squared between them, we have force given by k/x^2, where x is that distance and k is a constant. Then the (infinitesimal) work to move one of the objects the distance dx at the position x is given by $\frac{k}{x^2}\,dx$ (or, putting it roughly, force times distance). Next we note that the work required to pull the second object infinitely far from the first would be given by

$$\int_{s_0}^{\infty} \frac{k}{x^2}\,dx = \lim_{\beta\to\infty} \int_{s_0}^{\beta} \frac{k}{x^2}\,dx = \lim_{\beta\to\infty} \left[\frac{-k}{\beta} + \frac{k}{s_0}\right] = \frac{k}{s_0},$$

where $s_0 > 0$ is the original distance between them. Just as in Example 9.4.1 at the beginning of this section (page 689), we have that the integral is finite. This total work can be equated to the kinetic energy it would need to "escape" the stationary object, and from that kinetic energy $\frac{1}{2}mv^2$ (where $m =$ mass) we could find the necessary velocity of the second object for it to have that energy (ability to do the work) in kinetic form. The required velocity v is then called the *escape velocity.*

In the definition of derivative,

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

we needed a limit to "break through" the fact that we would really like Δx to *be* zero (but it can not). So too do we need limits to "break through" improper points in a definite integral, where otherwise the Fundamental Theorem of Calculus does not hold (because of a lack of continuity of the function, a key requirement of the theorem). For escape velocity, we have to break through the fact that you need to be able to compute how much energy is needed for the second object to move away from the first "forever," and not be recaptured. Other physical applications can also be found for improper integrals, but we will not pursue any more in this chapter.

9.4.5 A Complication With Oscillatory and Other Integrals

This subsection makes a point which is usually left to graduate-level real analysis courses (and so the reader should not be too anxious if it seems overly technical). This point has been hinted at previously, and is only included here for full disclosure. The point is that any integral is only said to converge if we can rearrange the order in which we count the signed area between the curve and the x-axis any way we like, and still come out with the same expression for total signed area.

So for instance, if an improper integral is to converge we should be able to add up all the positive area first, and then add up the negative area separately, and when we combine them we get a finite expression which is the same as if we found another way to sum the areas.

Consider again Example 9.4.7, page 697. There are infinitely many subintervals on which that function $e^{-x}\sin 4x$ is positive, and infinitely many on which it is negative. Our limit, however, has us sliding β towards infinity ($\beta \to \infty$), along the way allowing the areas counted to alternate between postive and negative, partially cancelling each other in turn as β increases. It is not clear at this point that, in fact, the positive areas alone sum to a finite number, as do the negative areas. This is equivalent[8] to the statement that it does not matter in which order the areas are summed, as long as any chosen area in the integral is eventually part of the sum; we will always get the same answer for $\int_0^\infty e^{-x}\sin 4x\,dx$.

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[8] That these two ideas are equivalent is a fact usually pondered for some time by the average mathematics graduate student.

A general theory can be proposed, but first a little notation is necessary. Rather than giving endpoints (a.k.a. *limits*) of integration, one notation has the integral written as "over the set." So for instance, the definite integral of $f(x)$ with respect to x over the interval $[a, b]$ is written

$$\int_a^b f(x)\,dx = \int_{[a,b]} f(x)\,dx.$$

If there is a function whose area is defined by disjoint intervals, this second notation is better. For instance, if we would like to find the combined area under $y = x^2$ for $-2 \le x \le -1$ and $1 \le x \le 2$, we write one of the following:

$$\int_{-2}^{-1} x^2\,dx + \int_1^2 x^2\,dx = \int_S x^2\,dx, \qquad \text{where } S = [-2,-1] \cup [1,2].$$

With this notation we can mention that the necessary and sufficient condition for an improper integral of a function $f(x)$ over a set S to converge is

$$\int_S |f(x)|\,dx < \infty. \tag{9.3}$$

Before explaining why this is reasonable, first we make two subintegrals:

$$\int_S |f(x)|\,dx = \int_{S_+} |f(x)|\,dx + \int_{S_-} |f(x)|\,dx \tag{9.4}$$

where $S_+ = \{x \mid f(x) \ge 0\}$ and $S_- = \{x \mid f(x) < 0\}$.

This integral $\int_S |f(x)|\,dx$ does not allow the positive and negative parts of the original integral to cancel: if the positive area is infinite, the total integral is greater than or equal to the integral over S_+, and is therefore infinite as well. Similarly with the negative part (though its sign is changed in (9.3) and (9.4)).

A function that satisfies (9.3) is called $L^1(S)$ (pronounced, "ell-one of S"), and will also satisfy that $\int_S f(x)\,dx$ converges, and we can use the limiting process for improper integrals if it is indeed improper. Now

$$\int_0^\infty \left|e^{-x}\sin 4x\right|\,dx \le \int_0^\infty e^{-x}\,dx = \lim_{\beta\to\infty}\int_0^\beta e^{-x}\,dx = \lim_{\beta\to\infty}\left[-e^{-\beta} + e^0\right] = 1 < \infty,$$

so $e^{-x}\sin 4x \in L^1([0,\infty))$, and we can compute the integral as we did in the example. However,

$$\int_1^\infty \frac{\sin x}{x^2}\,dx \qquad \text{converges, while} \int_1^\infty \frac{\sin x}{x}\,dx \qquad \text{diverges,}$$

though for both cases $\lim_{\beta\to\infty}\left(\int_1^\beta f(x)\,dx\right)$ exists, as we will see when we study "alternating series" later. The reason for the different outcomes is that the first integral's convergence does not rely on the sequential cancellation from the alternating (+/−) areas (but on the shrinking in absolute value of the integrand), where the second does.

It is beyond the scope of the text to pursue L^1-function theory further, but the same themes will come up for a time when we develop alternating series and absolute convergence.[9]

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[9] For reference, $f \in L^p(S)$ means $\displaystyle\int_S |f(x)|^p\,dx < \infty$. When $p = 1$ we have our previous definition of $L^1(S)$.