Geometry Problems

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1. Let M be the midpoint of the side AC of acute-angled triangle ABC with $AB > BC$. Let Ω be the circumcircle of ABC. The tangents to Ω at the points A and C meet at P, and BP and AC intersect at S. Let AD be the altitude of the triangle ABP and ω the circumcircle of the triangle CSD. Suppose ω and Ω intersect at $K \neq C$. Prove that $\angle CKM = 90°$. (All-Russian Olympiad 2014)

2. Given a triangle ABC with $AB > BC$, let Ω be the circumcircle. Let M, N lie on the sides AB, BC respectively, such that $AM = CN$. Let K be the intersection of MN and AC. Let P be the incentre of the triangle AMK and Q be the K-excentre of the triangle CNK. If R is midpoint of the arc ABC of Ω then prove that $RP = RQ$. (All-Russian Olympiad 2014)

3. Let M be the midpoint of the side AC of $\triangle ABC$. Let $P \in AM$ and $Q \in CM$ be such that $PQ = \frac{AC}{2}$. Let (ABQ) intersect with BC at $X \neq B$ and (BCP) intersect with BA at $Y \neq B$. Prove that the quadrilateral $BXMY$ is cyclic. (All-Russian Olympiad 2014)

4. Acute-angled triangle ABC is inscribed into circle Ω. Lines tangent to Ω at B and C intersect at P. Points D and E are on AB and AC such that PD and PE are perpendicular to AB and AC respectively. Prove that the orthocentre of triangle ADE is the midpoint of BC. (All-Russian Olympiad 2013)

5. Squares $CAKL$ and $CBMN$ are constructed on the sides of acute-angled triangle ABC, outside of the triangle. Line CN intersects line segment AK at X, while line CL intersects line segment BM at Y. Point P, lying inside triangle ABC, is an intersection of the circumcircles of triangles KXN and LYM. Point S is the midpoint of AB. Prove that angle $\angle ACS = \angle BCP$. (All-Russian Olympiad 2013)

6. Inside the inscribed quadrilateral $ABCD$ are marked points P and Q, such that $\angle PDC + \angle PCB$, $\angle PAB + \angle PBC$, $\angle QCD + \angle QDA$ and $\angle QBA + \angle QAD$ are all equal to $90°$. Prove that the line PQ has equal angles with lines AD and BC. (All-Russian Olympiad 2013)

7. The incircle of triangle ABC has centre I and touches the sides BC, CA, AB at points A_1, B_1, C_1, respectively. Let I_a, I_b, I_c be excentres of triangle ABC, touching the sides BC, CA, AB respectively. The segments I_aB_1 and I_bA_1 intersect at C_2. Similarly, segments I_bC_1 and I_cB_1 intersect at A_2, and the segments I_cA_1 and I_aC_1 at B_2. Prove that I is the center of the circumcircle of the triangle $A_2B_2C_2$. (All-Russian Olympiad 2013)

8. Let ω be the incircle of the triangle ABC and with centre I. Let Γ be the circumcircle of the triangle AIB. Circles ω and Γ intersect at the point X and Y. Let Z be the intersection of the common tangents of the circles ω and Γ. Show that the circumcircle of the triangle XYZ is tangent to the circumcircle of the triangle ABC. (All-Russian Olympiad 2013)

9. Consider the parallelogram $ABCD$ with obtuse angle A. Let H be the feet of perpendicular from A to the side BC. The median from C in triangle ABC meets the circumcircle of triangle ABC at the point K. Prove that points K, H, C, D lie on the same circle. (All-Russian Olympiad 2012)

10. The points A_1, B_1, C_1 lie on the sides sides BC, AC and AB of the triangle ABC respectively. Suppose that $AB_1 - AC_1 = CA_1 - CB_1 = BC_1 - BA_1$. Let I_A, I_B, I_C be the incentres of triangles AB_1C_1, A_1BC_1 and A_1B_1C respectively. Prove that the circumcentre of triangle $I_AI_BI_C$ is the incentre of triangle ABC. (All-Russian Olympiad 2012)

11. The inscribed circle ω of the non-isosceles acute-angled triangle ABC touches the side BC at the point D. Suppose that I and O are the centres of inscribed circle and circumcircle of triangle ABC respectively. The circumcircle of triangle ADI intersects AO at the points A and E. Prove that AE is equal to the radius r of ω. (All-Russian Olympiad 2012)

12. The point E is the midpoint of the segment connecting the orthocentre of the scalene triangle ABC and the point A. The incircle of triangle ABC incircle is tangent to AB and AC at points C' and B' respectively. Prove that point F, the point symmetric to point E with respect to line $B'C'$, lies on the line that passes through both

the circumcentre and the incentre of triangle ABC. (All-Russian Olympiad 2012)

13. The points A_1, B_1, C_1 lie on the sides BC, CA and AB of the triangle ABC respectively. Suppose that $AB_1 - AC_1 = CA_1 - CB_1 = BC_1 - BA_1$. Let O_A, O_B and O_C be the circumcentres of triangles AB_1C_1, A_1BC_1 and A_1B_1C respectively. Prove that the incentre of triangle $O_AO_BO_C$ is the incentre of triangle ABC too. (All-Russian Olympiad 2012)

14. Given is an acute angled triangle ABC. A circle going through B and the triangle's circumcenter, O, intersects BC and BA at points P and Q respectively. Prove that the intersection of the heights of the triangle POQ lies on line AC. (All-Russian Olympiad 2011)

15. Given is an acute triangle ABC. Its heights BB_1 and CC_1 are extended past points B_1 and C_1. On these extensions, points P and Q are chosen, such that angle PAQ is right. Let AF be a height of triangle APQ. Prove that angle BFC is a right angle. (All-Russian Olympiad 2011)

16. On side BC of parallelogram $ABCD$ (A is acute) lies point T so that triangle ATD is an acute triangle. Let O_1, O_2, and O_3 be the circumcenters of triangles ABT, DAT, and CDT respectively. Prove that the orthocenter of triangle $O_1O_2O_3$ lies on line AD. (All-Russian Olympiad 2011)

17. Let N be the midpoint of arc ABC of the circumcircle of triangle ABC, let M be the midpoint of AC and let I_1, I_2 be the incentres of triangles ABM and CBM. Prove that points I_1, I_2, B, N lie on a circle. (All-Russian Olympiad 2011)

18. Lines tangent to circle O in points A and B, intersect in point P. Point Z is the center of O. On the minor arc AB, point C is chosen not on the midpoint of the arc. Lines AC and PB intersect at point D. Lines BC and AP intersect at point E. Prove that the circumcentres of triangles ACE, BCD, and PCZ are collinear. (All-Russian Olympiad 2010)

19. Let O be the circumcentre of the acute non-isosceles triangle ABC. Let P and Q be points on the altitude AD such that OP and OQ are perpendicular to AB and AC respectively. Let M be the midpoint of BC and S be the circumcentre of triangle OPQ. Prove that $\angle BAS = \angle CAM$. (All-Russian Olympiad 2010)

20. Into triangle ABC gives point K lies on bisector of $\angle BAC$. Line CK intersect circumcircle ω of triangle ABC at $M \neq C$. Circle Ω passes through A, touch CM at K and intersect segment AB at $P \neq A$ and ω at $Q \neq A$. Prove, that P, Q, M lies at one line. (All-Russian Olympiad 2010)

21. Quadrilateral $ABCD$ is inscribed into circle ω, AC intersect BD in point K. Points M_1, M_2, M_3, M_4-midpoints of arcs AB, BC, CD, and DA respectively. Points I_1, I_2, I_3, I_4-incenters of triangles ABK, BCK, CDK, and DAK respectively. Prove that lines M_1I_1, M_2I_2, M_3I_3, and M_4I_4 all intersect in one point. (All-Russian Olympiad 2010)

22. Let be given a triangle ABC and its internal angle bisector BD ($D \in BC$). The line BD intersects the circumcircle Ω of triangle ABC at B and E. Circle ω with diameter DE cuts Ω again at F. Prove that BF is the symmedian line of triangle ABC. (All-Russian Olympiad 2009)

23. The incircle (I) of a given scalene triangle ABC touches its sides BC, CA, AB at A_1, B_1, C_1, respectively. Denote ω_B, ω_C the incircles of quadrilaterals BA_1IC_1 and CA_1IB_1, respectively. Prove that the internal common tangent of ω_B and ω_C different from IA_1 passes through A. (All-Russian Olympiad 2009)

24. Let be given a parallelogram $ABCD$ and two points A_1, C_1 on its sides AB, BC, respectively. Lines AC_1 and CA_1 meet at P. Assume that the circumcircles of triangles AA_1P and CC_1P intersect at the second point Q inside triangle ACD. Prove that $\angle PDA = \angle QBA$. (All-Russian Olympiad 2009)

25. In a scalene triangle ABC, H and M are the orthocenter an centroid respectively. Consider the triangle formed by the lines through A, B and C perpendicular to AM, BM and CM respectively. Prove that the centroid of this triangle lies on the line MH. (All-Russian Olympiad 2008)

26. A circle ω with center O is tangent to the rays of an angle BAC at B and C. Point Q is taken inside the angle

BAC. Assume that point P on the segment AQ is such that $AQ \perp OP$. The line OP intersects the circumcircles ω_1 and ω_2 of triangles BPQ and CPQ again at points M and N. Prove that $OM = ON$. (All-Russian Olympiad 2008)

27. In a scalene triangle ABC the altitudes AA_1 and CC_1 intersect at H, O is the circumcenter, and B_0 the midpoint of side AC. The line BO intersects side AC at P, while the lines BH and A_1C_1 meet at Q. Prove that the lines HB_0 and PQ are parallel. (All-Russian Olympiad 2008)

28. In convex quadrilateral $ABCD$, the rays BA, CD meet at P, and the rays BC, AD meet at Q. H is the projection of D on PQ. Prove that there is a circle inscribed in $ABCD$ if and only if the incircles of triangles ADP, CDQ are visible from H under the same angle. (All-Russian Olympiad 2008)

29. Given a rhombus $ABCD$. A point M is chosen on its side BC. The lines, which pass through M and are perpendicular to BD and AC, meet line AD in points P and Q respectively. Suppose that the lines PB, QC, AM have a common point. Find all possible values of a ratio $\frac{BM}{MC}$. (All-Russian Olympiad 2007)

30. A line, which passes through the incentre I of the triangle ABC, meets its sides AB and BC at the points M and N respectively. The triangle BMN is acute. The points K, L are chosen on the side AC such that $\angle ILA = \angle IMB$ and $\angle KC = \angle INB$. Prove that $AM + KL + CN = AC$. (All-Russian Olympiad 2007)

31. BB_1 is a bisector of an acute triangle ABC. A perpendicular from B_1 to BC meets a smaller arc BC of a circumcircle of ABC in a point K. A perpendicular from B to AK meets AC in a point L. BB_1 meets arc AC in T. Prove that K, L, T are collinear. (All-Russian Olympiad 2007)

32. Two circles ω_1 and ω_2 intersect in points A and B. Let PQ and RS be segments of common tangents to these circles (points P and R lie on ω_1, points Q and S lie on ω_2). It appears that $RB \parallel PQ$. Ray RB intersects ω_2 in a point $W \neq B$. Find RB/BW. (All-Russian Olympiad 2007)

33. The incircle of triangle ABC touches its sides BC, AC, AB at the points A_1, B_1, C_1 respectively. A segment AA_1 intersects the incircle at the point $Q \neq A_1$. A line ℓ through A is parallel to BC. Lines A_1C_1 and A_1B_1 intersect ℓ at the points P and R respectively. Prove that $\angle PQR = \angle B_1QC_1$. (All-Russian Olympiad 2007)

34. Given a triangle ABC. Let a circle ω touch the circumcircle of triangle ABC at the point A, intersect the side AB at a point K, and intersect the side BC. Let CL be a tangent to the circle ω, where the point L lies on ω and the segment KL intersects the side BC at a point T. Show that the segment BT has the same length as the tangent from the point B to the circle ω. (All-Russian Olympiad 2006)

35. Let P, Q, R be points on the sides AB, BC, CA of a triangle ABC such that $AP = CQ$ and the quadrilateral $RPBQ$ is cyclic. The tangents to the circumcircle of triangle ABC at the points C and A intersect the lines RQ and RP at the points X and Y, respectively. Prove that $RX = RY$. (All-Russian Olympiad 2006)

36. Consider an isosceles triangle ABC with $AB = AC$, and a circle ω which is tangent to the sides AB and AC of this triangle and intersects the side BC at the points K and L. The segment AK intersects the circle ω at a point M (apart from K). Let P and Q be the reflections of the point K in the points B and C, respectively. Show that the circumcircle of triangle PMQ is tangent to the circle ω. (All-Russian Olympiad 2006)

37. Let K and L be two points on the arcs AB and BC of the circumcircle of a triangle ABC, respectively, such that $KL \parallel AC$. Show that the incenters of triangles ABK and CBL are equidistant from the midpoint of the arc ABC of the circumcircle of triangle ABC. (All-Russian Olympiad 2006)

38. Given a triangle ABC. The angle bisectors of the angles ABC and BCA intersect the sides CA and AB at the points B_1 and C_1, and intersect each other at the point I. The line B_1C_1 intersects the circumcircle of triangle ABC at the points M and N. Prove that the circumradius of triangle MIN is twice as long as the circumradius of triangle ABC. (All-Russian Olympiad 2006)

39. Given a parallelogram $ABCD$ with $AB < BC$, show that the circumcircles of the triangles APQ share a second common point (apart from A) as P, Q move on the sides BC, CD respectively s.t. $CP = CQ$. (All-Russian Olympiad 2005)

40. w_B and w_C are excircles of a triangle ABC. The circle w'_B is symmetric to w_B with respect to the midpoint of AC, the circle w'_C is symmetric to w_C with respect to the midpoint of AB. Prove that the radical axis of w'_B and w'_C halves the perimeter of ABC. (All-Russian Olympiad 2005)

41. We have an acute-angled triangle ABC, and AA', BB' are its altitudes. A point D is chosen on the arc ACB of the circumcircle of ABC. If $P = AA' \cap BD, Q = BB' \cap AD$, show that the midpoint of PQ lies on $A'B'$. (All-Russian Olympiad 2005)

42. Let A', B', C' be points, in which excircles touch corresponding sides of triangle ABC. Circumcircles of triangles $A'B'C$, $AB'C'$, $A'BC'$ intersect a circumcircle of ABC in points $C_1 \neq C$, $A_1 \neq A$, $B_1 \neq B$ respectively. Prove that a triangle $A_1B_1C_1$ is similar to a triangle, formed by points, in which incircle of ABC touches its sides. (All-Russian Olympiad 2005)

43. A quadrilateral $ABCD$ without parallel sides is circumscribed around a circle with centre O. Prove that O is a point of intersection of middle lines of quadrilateral $ABCD$ (i.e. barycentre of points A, B, C, D) iff $OA \cdot OC = OB \cdot OD$. (All-Russian Olympiad 2005)

44. Let $ABCD$ be a circumscribed quadrilateral (i. e. a quadrilateral which has an incircle). The exterior angle bisectors of the angles DAB and ABC intersect each other at K; the exterior angle bisectors of the angles ABC and BCD intersect each other at L; the exterior angle bisectors of the angles BCD and CDA intersect each other at M; the exterior angle bisectors of the angles CDA and DAB intersect each other at N. Let K_1, L_1, M_1 and N_1 be the orthocenters of the triangles ABK, BCL, CDM and DAN, respectively. Show that the quadrilateral $K_1L_1M_1N_1$ is a parallelogram. (All-Russian Olympiad 2004)

45. Let O be the circumcenter of an acute-angled triangle ABC, let T be the circumcenter of the triangle AOC, and let M be the midpoint of the segment AC. We take a point D on the side AB and a point E on the side BC that satisfy $\angle BDM = \angle BEM = \angle ABC$. Show that the straight lines BT and DE are perpendicular. (All-Russian Olympiad 2004)

46. Let the incircle of the quadrilateral $ABCD$ touch its sides AB, BC, CD, and DA in the points K, L, M, and N, respectively. The exterior angle bisectors of the angles DAB and ABC intersect each other at a point K'. The exterior angle bisectors of the angles ABC and BCD intersect each other at a point L'. The exterior angle bisectors of the angles BCD and CDA intersect each other at a point M'. The exterior angle bisectors of the angles CDA and DAB intersect each other at a point N'. Prove that the straight lines KK', LL', MM', and NN' are concurrent. (All-Russian Olympiad 2004)

47. Let $I(A)$ and $I(B)$ be the centers of the excircles of a triangle ABC, which touches the sides BC and CA in its interior. Furthermore let P a point on the circumcircle ω of the triangle ABC. Show that the center of the segment which connects the circumcenters of the triangles $I(A)CP$ and $I(B)CP$ coincides with the center of the circle ω. (All-Russian Olympiad 2004)