COMBINATORICS

Stephan Wagner

Version: December 2013

Contents

Preface

These notes are aimed at advanced participants in mathematical olympiads and their coaches. Some of the parts cover more than what is usually needed in mathematical competitions. For example, the parts of Chapter 2 that follow Corollary 2.4 or the treatment of generating functions in Section 4.3 are mostly aimed at particularly interested learners. As far as graph theory (Chapter 7) is concerned, it should be mentioned that general understanding of the main concepts is more important for the solution of olympiad problems than the actual theory that is usually not needed at all.

Any comments, suggestions, corrections, etc. can be directed to me via e-mail:

`swagner@sun.ac.za`

I wish everyone a pleasant journey through the world of combinatorics, and I hope that you will find these notes useful.

Chapter 1

Elementary enumeration principles

Sequences

Theorem 1.1 *There are n^k different sequences of length k that can be formed from elements of a set X consisting of n elements (elements are allowed to occur several times in a sequence).*

Proof: For every element of the sequence, we have exactly n choices. Therefore, there are

$$\underbrace{n \cdot n \cdot \ldots \cdot n}_{k \text{ times}} = n^k$$

different possibilities. □

Example: Given an "alphabet" of n letters, there are exactly n^k k-letter words. For instance, there are 8 three-digit words (not necessarily meaningful) that can be formed from the letters S and O:

SSS, SSO, SOS, OSS, SOO, OSO, OOS, OOO.

The number of 100-letter words over the alphabet A,C,G,T is 4^{100}, which is a 61-digit number; DNA strings as they occur in cells of living organisms are much longer, of course...

Permutations

Theorem 1.2 *The number of possibilities to arrange n (distinguishable) objects in a row (so-called* permutations*) is*

$$n! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1.$$

Proof: There are obviously n choices for the first position, then $n-1$ remaining choices for the second position (as opposed to the previous theorem), $n-2$ for the third position, etc. Therefore, one obtains the stated formula. □

Example: There are 6 possibilities to arrange the letters A,E,T in a row:

AET, ATE, EAT, ETA, TAE, TEA.

REMARK: By definition, $n!$ satisfies the equation $n! = n \cdot (n-1)!$, which remains true if one defines $0! = 1$ (informally, there is exactly one possibility to arrange 0 objects, and that is to do nothing at all).

EXAMPLE: In how many ways can eight rooks be placed on an 8×8-chessboard in such a way that no horizontal or vertical row contains two rooks?

In order to solve this problem, let us assign coordinates (a-h and 1-8 respectively) to the squares of the chessboard. A possible configuration would then be a3, b5, c1, d8, e6, f2, g4, h7 (for instance). Generally, there must be exactly one rook on each vertical row (a-h), and analogously one rook on each horizontal row (1-8). Each permutation of the numbers 1 to 8 corresponds to exactly one feasible configuration (in the above case, 3-5-1-8-6-2-4-7), and so there are exactly $8! = 40320$ possibilities.

Sequences without repetitions

Theorem 1.3 *The number of sequences of length k without repetitions whose elements are taken from a set X comprising n elements is*

$$n^{\underline{k}} = n \cdot (n-1) \cdot (n-2) \cdot \ldots (n-k+1) = \frac{n!}{(n-k)!}.$$

Proof: The proof is essentially the same as for Theorem 1.2: for the first element, there are n possible choices, then $n-1$ for the second element, etc. For the last element, there are $n-k+1$ choices left. □

REMARK: The case of permutations is clearly a special case of Theorem 1.3, corresponding to $k = n$. $n^{\underline{k}}$ is called a *falling factorial* (read: "n to the k falling").

Choosing a subset

Theorem 1.4 *The number of possibilities to choose a subset of k elements from a set of n elements (the order being irrelevant) is*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Proof: Let x be the number of possibilities that we are looking for. Once the k elements have been chosen (for which there are x possible ways), one has $k!$ possibilities (by Theorem 1.2) to arrange them in a sequence. Therefore, $x \cdot k!$ is exactly the number of possible sequences of k distinct elements, for which we have the formula

$$x \cdot k! = \frac{n!}{(n-k)!}$$

by Theorem 1.3, so that x is obtained immediately. □

REMARK: The difference between Theorem 1.3 and Theorem 1.4 lies in the fact that the order plays a role in the former, which is does not in the latter. Each subset corresponds to exactly $k!$ sequences: for instance, the subset $\{A, E, T\}$ of the set $\{A, B, \ldots, Z\}$ corresponds to the sequences

AET, ATE, EAT, ETA, TAE, TEA.

The formula for the binomial coefficient only makes sense if $0 \leq k \leq n$. This is also quite intuitive as no subset can comprise more elements than the original set. It is often useful to define $\binom{n}{k} = 0$ if either $k < 0$ or $k > n$. Later we will also give a more general definition for the binomial coefficients.

EXAMPLE: The number of six-element subsets of $\{1, 2, \ldots, 49\}$ (Lotto) is

$$\binom{49}{6} = \frac{49!}{6! \cdot 43!} = 13983816.$$

REMARK: An obvious property of $\binom{n}{k}$ is the identity

$$\binom{n}{k} = \binom{n}{n-k},$$

which follows immediately from the formula. However, it also has a combinatorial meaning: choosing k elements is equivalent to not choosing $n - k$ elements. The generalisation of this principle leads us to the so-called *multinomial coefficient*.

Dividing a set into groups

Theorem 1.5 *The number of possibilities to divide a set X into groups $X_1, X_2, \ldots, X_r$ whose sizes are prescribed to be $k_1, k_2, \ldots, k_r$ respectively (where $k_1 + k_2 + \ldots + k_r = n$) is given by*

$$\binom{n}{k_1, k_2, \ldots, k_r} = \frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_r!}.$$

Proof: By induction on r; for $r = 1$, the statement is trivial. For $r \geq 2$, one has $\binom{n}{k_1}$ choices for the elements of X_1, and by the induction hypothesis,

$$\frac{(n-k_1)!}{k_2! k_3! \ldots k_r!}$$

possible ways to divide the remaining $n - k_1$ elements. Therefore, we have exactly

$$\binom{n}{k_1} \cdot \frac{(n-k_1)!}{k_2! k_3! \ldots k_r!} = \frac{n!}{k_1!(n-k_1)!} \cdot \frac{(n-k_1)!}{k_2! k_3! \ldots k_r!} = \frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_r!}$$

possibilities, as claimed. □

Choosing a multiset

Theorem 1.6 *The number of ways to choose k elements from a set of n elements, repetitions allowed, is $\binom{n+k-1}{k}$.*

Proof: Let $X = \{x_1, x_2, \ldots, x_n\}$ be the set. A choice is characterised by the number of times that each of the elements is selected. If l_i denotes the multiplicity of x_i in our collection, then the problem is equivalent to determining the number of solutions of

$$l_1 + l_2 + \ldots + l_n = k,$$

where $l_1, l_2, \ldots, l_n$ have to be non-negative integers. Equivalently, we can write $m_i = l_i + 1$ and ask for the number of positive integer solutions to the equation

$$m_1 + m_2 + \ldots + m_n = k + n. \tag{1.1}$$

Let us imagine $k + n$ dots in a row. Each solution to the equation (1.1) corresponds to a way of separating the dots by inserting $n - 1$ bars at certain places (Figure 1.1). Since there are $n + k - 1$ positions for the bars, one has

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

possible ways to place the bars. □

$$\bullet \mid \bullet\ \bullet\ \bullet\ \bullet \mid \bullet\ \bullet\ \bullet \mid \bullet \mid \bullet\ \bullet$$

Figure 1.1: Dots and bars.

Remark: A finite sequence $m_1, m_2, \ldots, m_k$ of positive integers summing to n (that is, $n = m_1 + m_2 + \ldots + m_k$) is called a *composition* of n; the above argument ("dots and bars") shows that every positive integer n has exactly $\binom{n-1}{k-1}$ compositions into k summands.

1.1 Problems

Problem 1 How many Lotto-combinations (6 numbers out of $\{1, 2, \ldots, 49\}$) contain two consecutive numbers?

Problem 2 King Arthur chooses three of the 25 knights sitting around his table to fight a fearsome dragon. How many possible choices are there, if no two of the chosen knights should sit next to each other?

Problem 3 How many possible ways are there to form five-letter words using only the letters A,B,C,D,E,F,G,H? How many such words are there that do not contain a letter twice?

Problem 4 In how many possible orders can the letters of the word MATHEMATICS be arranged?

Problem 5 At a sokkie, there are 20 girls and 20 boys. How many ways are there to form 20 pairs?

Chapter 2

Properties of binomial coefficients, combinatorial identities

It has already been mentioned that $\binom{n}{k} = \binom{n}{n-k}$. This is just one of a huge number of identities that are satisfied by the binomial coefficients. Some of them are presented here—mostly because the proofs are instructive and the methods can be used frequently in different contexts.

2.1 The recursion

Theorem 2.1 *The binomial coefficients satisfy the recursion*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad (0 \le k \le n).$$

Proof: The identity can be verified easily as follows:

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\ &= \frac{k \cdot (n-1)!}{k!(n-k)!} + \frac{(n-k) \cdot (n-1)!}{k!(n-k)!} \\ &= \frac{n \cdot (n-1)!}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.\end{aligned}$$

On the other hand, one can also argue as follows: let X be an element of a set M with n elements. If one wants to choose k elements from M, one can first decide whether X should be part of this selection or not. In the former case, one has to choose $k-1$ elements from the remaining $n-1$ elements ($\binom{n-1}{k-1}$ possibilities), in the latter case one still has to choose k elements ($\binom{n-1}{k}$ possibilities). Hence we obtain the identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

as it was claimed. $\square$

Note that this equation remains true if one sets $\binom{n}{k} = 0$ for $k > n$ or $k < 0$. A good way to represent this recursion graphically is to write down the values of the binomial coefficients $\binom{n}{k}$ in *Pascal's triangle*: the n-th row consists of the values $\binom{n}{k}$ $(0 \leq k \leq n)$:

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&&\\
&&&&&&&1&&1&&&&&&&\\
&&&&&&1&&2&&1&&&&&&\\
&&&&&1&&3&&3&&1&&&&&\\
&&&&1&&4&&6&&4&&1&&&&\\
&&&1&&5&&10&&10&&5&&1&&&\\
&&1&&6&&15&&20&&15&&6&&1&&\\
&1&&7&&21&&35&&35&&21&&7&&1&\\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}
$$

One notices immediately that each number is the sum of the two numbers above it. Moreover, it is possible to define $\binom{\alpha}{k}$ for arbitrary real numbers α, namely as

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!}.$$

Theorem 2.1 remains correct with this definition.

2.2 The binomial theorem and its applications

One of the most important theorems in connection with binomial coefficients is the *binomial theorem*:

Theorem 2.2 (Binomial theorem) *For all integers $n \geq 0$, the identity*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

holds.

Proof: We prove the statement by means of induction: for $n = 0$, the left hand side is $(x+y)^0 = 1$, while the right hand side is $\binom{0}{0} x^0 y^0 = 1$. Hence the assertion is true in this case. Now assume that the identity holds for a certain n. We exploit the definition $\binom{n}{n+1} = \binom{n}{-1} = 0$ and theorem 2.1 to obtain

$$(x+y)^{n+1} = (x+y) \cdot (x+y)^n = (x+y) \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^{k} y^{n-k+1} \\
&= \sum_{k=1}^{n+1} \binom{n}{k-1} x^{k} y^{n-k+1} + \sum_{k=0}^{n} \binom{n}{k} x^{k} y^{n-k+1} \\
&= \sum_{k=0}^{n+1} \binom{n}{k-1} x^{k} y^{n-k+1} + \sum_{k=0}^{n+1} \binom{n}{k} x^{k} y^{n-k+1} \\
&= \sum_{k=0}^{n+1} \left(\binom{n}{k-1} + \binom{n}{k} \right) x^{k} y^{n+1-k} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{k} y^{n+1-k},
\end{aligned}
$$

which shows that the statement is true for $n+1$. This completes the induction.

A more combinatorial approach to the binomial theorem would be as follows: if the expression $(x+y)^n$ is multiplied out, one obtains all possible products of x and y that can be formed by choosing one summand from each of the n factors. We obtain a product $x^k y^{n-k}$ if and only if x is selected k times and y is selected $n-k$ times. By Theorem 1.4 there are exactly $\binom{n}{k}$ possibilities to achieve this, hence the expression $x^k y^{n-k}$ occurs exactly $\binom{n}{k}$ times in the expansion. □

Remark: It should be mentioned that one usually sets $0^0 = 1$ in this context, so that the binomial theorem remains true for $y = 0$ (and/or $x = 0$).

One can obtain a number of identities directly from the binomial theorem, in particular the following:

Theorem 2.3 *The following formulas for sums of binomial coefficients hold:*

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$$

for $n \geq 0$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \ldots + (-1)^n \binom{n}{n} = 0$$

for $n > 0$,

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} = \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k} = 2^{n-1}$$

for $n > 0$.

Proof: The first equation is obtained by putting $x = y = 1$ in the binomial theorem, the second equation for $x = 1$ and $y = -1$. The third equation follows by adding (subtracting, respectively) the first two equations and noticing that

$$1 + (-1)^k = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases} \text{ resp. } 1 - (-1)^k = \begin{cases} 0 & k \text{ even} \\ 2 & k \text{ odd} \end{cases}$$

holds. □

The first equation allows the following interpretation: since $\binom{n}{k}$ is exactly the number of subsets of an n-element set with precisely k elements, the sum on the left hand side is exactly the number of all subsets of an n-element set. Hence we have

Corollary 2.4 An n-element set has exactly 2^n distinct subsets.

Example: The set {A,B,C} has the 8 subsets

$$\emptyset = \{\}, \ \{\text{A}\}, \ \{\text{B}\}, \ \{\text{C}\}, \ \{\text{A,B}\}, \ \{\text{A,C}\}, \ \{\text{B,C}\}, \ \{\text{A,B,C}\}.$$

Remark: Of course it is possible to obtain Corollary 2.4 without using the binomial theorem. In selecting a subset, one has two possibilities for each of the elements: it can be an element of the subset or not. Hence the number of possible subsets is exactly $2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$. On the other hand, one can also deduce from Theorem 2.3 that a finite, non-empty set has the same number of subsets of even cardinality and of odd cardinality (which can also be proved combinatorially—this is left to the reader).

Problem 6 Prove for all positive integers n:

$$\sum_{k=0}^{n} \binom{3n}{3k} = \frac{1}{3}\left(2^{3n} + 2 \cdot (-1)^n\right).$$

Solution: We would like to use the trick that was applied in the proof of Theorem 2.3. However, the values 1 and -1 have to be replaced by something else. Let $\zeta_{1,2} = \frac{-1 \pm i\sqrt{3}}{2}$ be the third roots of unity, i.e., $\zeta_1^3 = \zeta_2^3 = 1$. Then we have $\zeta_1 + \zeta_2 + 1 = 0$, $\zeta_1^2 = \zeta_2$ and $\zeta_2^2 = \zeta_1$. If we plug $y = 1$ and $x = 1$, $x = \zeta_1$, $x = \zeta_2$ into the binomial theorem and add the equations, we obtain

$$\sum_{l=0}^{3n} \binom{3n}{l} \left(1 + \zeta_1^l + \zeta_2^l\right) = 2^{3n} + (1 + \zeta_1)^{3n} + (1 + \zeta_2)^{3n}.$$

The expression $1 + \zeta_1^l + \zeta_2^l$ takes on different values depending on l: for $l = 3k$, one has

$$1 + \zeta_1^l + \zeta_2^l = 1 + 1^k + 1^k = 3,$$

for $l = 3k + 1$,

$$1 + \zeta_1^l + \zeta_2^l = 1 + 1^k\zeta_1 + 1^k\zeta_2 = 1 + \zeta_1 + \zeta_2 = 0,$$

and for $l = 3k + 2$,

$$1 + \zeta_1^l + \zeta_2^l = 1 + 1^k\zeta_2 + 1^k\zeta_1 = 1 + \zeta_2 + \zeta_1 = 0,$$

hence

$$\sum_{k=0}^{n} 3\binom{3n}{3k} = 2^{3n} + (1+\zeta_1)^{3n} + (1+\zeta_2)^{3n}.$$

Now $1 + \zeta_{1,2} = \frac{1\pm\sqrt{3}i}{2}$ are sixth roots of unity, i.e., $(1+\zeta_{1,2})^3 = -1$ and $(1+\zeta_{1,2})^6 = 1$. Hence it follows immediately that

$$\sum_{k=0}^{n} \binom{3n}{3k} = \frac{1}{3}\left(2^{3n} + 2\cdot(-1)^n\right).$$

□

Another important sum formula is the following:

Theorem 2.5 (Vandermonde identity) *For integers $N, M, n \geq 0$, the following identity holds:*

$$\sum_{k=0}^{n} \binom{N}{k}\binom{M}{n-k} = \binom{N+M}{n}.$$

Proof: Sums of the form

$$\sum_{k=0}^{n} a_k b_{n-k}$$

are reminiscent of polynomial multiplication: If two polynomials $A(x) = \sum_{k=0}^{N} a_k x^k$ and $B(x) = \sum_{l=0}^{M} b_l x^l$ are given, then one obtains for the product of the two

$$A(x)\cdot B(x) = \sum_{k=0}^{N}\sum_{l=0}^{M} a_k b_l x^{k+l}.$$

The coefficient of x^n is obtained from those terms for which $k + l = n$, or equivalently $l = n - k$. It follows that the coefficient of x^n equals

$$\sum_{k=0}^{n} a_k b_{n-k},$$

where we set $a_k = 0$ if the degree N of the polynomial A is less than k (and likewise $b_l = 0$ if $l > M$). The same argument is also used later in Section 4.3.

In the present case the problem can be solved by applying the binomial theorem to the polynomial $(1+x)^{N+M}$, which is also the product of the two polynomials $(1+x)^N$ and $(1+x)^M$:

$$(1+x)^{N+M} = (1+x)^N \cdot (1+x)^M = \left(\sum_{k=0}^{N}\binom{N}{k}x^k\right)\left(\sum_{l=0}^{M}\binom{M}{l}x^l\right)$$

$$= \sum_{k=0}^{N} \sum_{l=0}^{M} \binom{N}{k} \binom{M}{l} x^{k+l} = \sum_{n=0}^{N+M} \sum_{k=0}^{n} \binom{N}{k} \binom{M}{n-k} x^n.$$

Now we obtain the desired identity upon comparing coefficients with

$$(1+x)^{N+M} = \sum_{n=0}^{N+M} \binom{N+M}{n} x^n.$$

Once again, it is possible to prove the identity by means of a simple counting argument: consider a set of $N+M$ elements, and divide it into two parts with N and M elements respectively. If one wants to select n elements, one can do so by first choosing k elements from the first part ($k \leq n$) and the remaining $n-k$ elements from the second part. For each k, there are precisely $\binom{N}{k}\binom{M}{n-k}$ possibilities. Summing over all k, we obtain the above expression for $\binom{N+M}{n}$. $\square$

The following special case of the Vandermonde identity is remarkable:

Corollary 2.6

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} = \binom{2n}{n}.$$

Problem 7 Prove that the following identity holds:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = \begin{cases} (-1)^{n/2} \binom{n}{n/2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Solution: Once again, we compare coefficients, namely in the identity

$$\left(\sum_{k=0}^{n} \binom{n}{n-k} x^{n-k} \right) \left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k \right) = (1+x)^n (1-x)^n = (1-x^2)^n = \sum_{k=0}^{n} \binom{n}{k} (-x^2)^k.$$

The coefficient of x^n on the right hand side is indeed $(-1)^{n/2}\binom{n}{n/2}$ if n is even and 0 otherwise. The coefficient on the left hand side is obtained by multiplying out:

$$\sum_{k=0}^{n} \binom{n}{n-k} \cdot (-1)^k \binom{n}{k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2.$$

$\square$

2.3 Bijections and the Catalan numbers

Various counting problems lead to the *Catalan numbers* that are presented briefly in this section to illustrate some important combinatorial ideas. First of all, let us consider the middle column in Pascal's triangle, namely the numbers $1, 2, 6, 20, 70, 252, \ldots$ given by the formula $\binom{2n}{n}$. One notices that they have the following remarkable property: they are divisible by $1, 2, 3, 4, 5, 6, \ldots$! In other words, the expression

$$\frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}$$

is always an integer, which is not at all obvious. It is a good exercise to prove this fact entirely by number-theoretic arguments.

We are interested in the combinatorial interpretation of the resulting integer sequence $1, 1, 2, 5, 14, 42, \ldots$. Let us consider the following problem:

Problem 8 Twenty children are queueing for ice cream that is sold at R5 per cone. Ten of the children have a R5 coin, the others want to pay with a R10 bill. At the beginning, the ice cream man does not have any change. How many possible arrangements of the twenty kids in a queue are there so that the ice cream man will never run out of change?

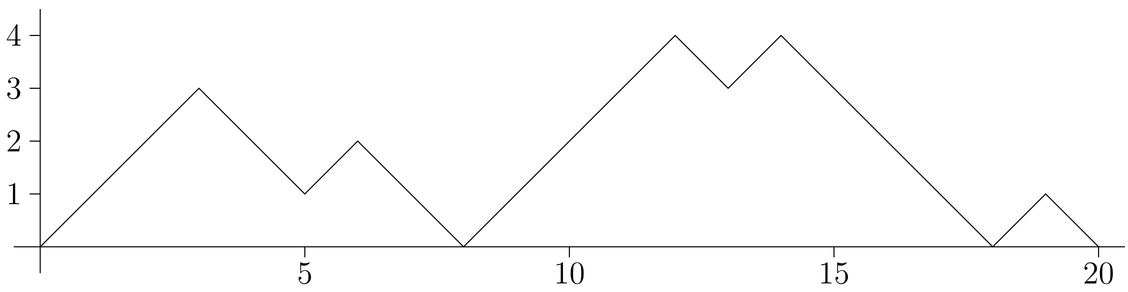

Figure 2.1: Diagram of the amount of change left.

Solution: Let us consider a diagram in which the amount of change left after each child is shown. If a child pays with a coin, the amount increases by 1, otherwise it decreases by 1. Our requirement is equivalent to the condition that the amount of change stays nonnegative throughout the process. The diagram thus forms some kind of "mountain range" of up- and down-steps (see Figure 2.1).

We add an additional up-step in the beginning and join several copies to obtain a periodic pattern whose lowest points are where the copies are "glued together" (Figure 2.2). Conversely, one can assign a feasible change diagram to each possible periodic pattern of 11 up- and 10 down-steps. To this end, draw a line whose slope is $\frac{1}{21}$ in such a way that it touches the periodic pattern in its lowest points, as shown in Figure 2.2. This divides the pattern uniquely into periods of length 21, and the last 20 steps of each period form a feasible change diagram.

We have thus found a *bijection* between change diagrams and periodic patterns, i.e., a $1-1$-correspondence that associates a change diagram with a periodic pattern and vice versa. It is clear now that their numbers have to be the same.

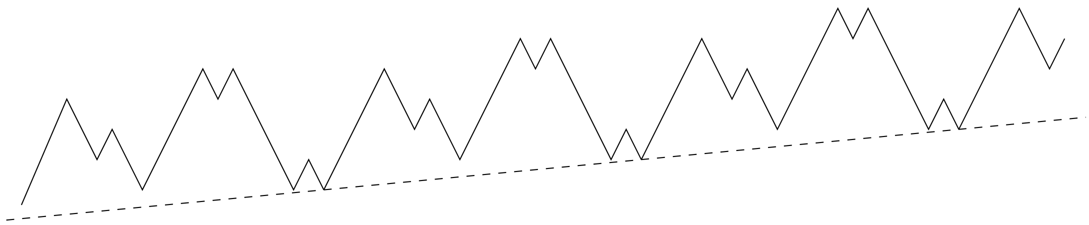

Figure 2.2: The associated periodic pattern.

There are $\binom{21}{10}$ possible periods for a periodic pattern (the number of ways to choose the positions of the 10 down-steps). However, each given pattern has 21 distinct periods (beginning with the first, second, ..., 21-st step), hence there are

$$\frac{1}{21}\binom{21}{10} = \frac{1}{21} \cdot \frac{21!}{10!11!} = \frac{20!}{10!}11! = \frac{1}{11}\binom{20}{10}$$

different periodic patterns.

More generally, if there are $2n$ kids queuing, one obtains the so-called *Catalan numbers* $\frac{1}{n+1}\binom{2n}{n}$, which also satisfy a certain recursion. We will encounter them again in Section 4.3. Since there are 10! ways to arrange the children with a R5 coin as well as 10! ways to arrange the children with a R10 bill, the answer to our original question is $10!^2 \cdot \frac{1}{11}\binom{20}{10} = \frac{1}{11} \cdot 20!$ (i.e., the probability that a random arrangement works out is $\frac{1}{11}$).

2.4 Additional problems

Problem 9 Prove the identity

$$\sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j} = 3^n.$$

Problem 10 How many words of length n can be formed from the letters A,B,C, such that the number of A's is even?

Problem 11 A single piece is placed on the lower left corner square of an $n \times n$-chessboard. The piece may only move horizontally or vertically, one square at a time. How many possible ways are there to move the piece to the opposite corner in $2(n-1)$ moves (the smallest possible number of moves)?

Problem 12 (IMO 1981) Let $1 \leq r \leq n$. We consider all r-element subsets of $\{1, \ldots, n\}$. Each of them has a minimum. Prove that the average of these minima is $\frac{n+1}{r+1}$.

Problem 13 (IMO Shortlist 2002) Let n be a positive integer. A sequence of n positive integers (not necessarily distinct) is called *full* if it satisfies the following condition:

for each positive integer $k \geq 2$, if the number k appears in the sequence then so does the number $k-1$, and moreover the first occurrence of $k-1$ comes before the last occurrence of k. For each n, how many full sequences are there?

Chapter 3

The principle of inclusion and exclusion

3.1 A simple example

A frequently occurring problem is to determine the size of the union or intersection of a number of sets, as in the following example:

Problem 14 At a certain university, all second-year science students may choose either mathematics, or physics, or both. The mathematics course is attended by 50 students, the physics course by 30 students. 15 students attend both courses. How many second-year science students are there?

Solution: Let M be the set of students taking mathematics and P the set of all students who take physics. By our conditions, the set of all students is the union $S = M \cup P$. If we add the sizes of the two sets, all students who attend both courses are counted twice. Therefore, we have to subtract the size of $M \cap P$, which yields the formula

$$|S| = |M \cup P| = |M| + |P| - |M \cap P|,$$

where $|X|$ is the number of elements of a set X. Plugging in, we find that there are $50 + 30 - 15 = 65$ students. $\square$

This principle can be generalised to unions (or intersections) of an arbitrary number of sets. Before we discuss the general formula, let us extend Example 14:

Problem 15 Third-year science students also have the opportunity to attend chemistry, but every student has to take at least one of the three courses. Altogether, there are 40 students in the mathematics class, 25 who attend physics, and 20 who attend chemistry. Furthermore, we know that 10 students do both mathematics and physics, 8 both mathematics and chemistry, and 7 physics and chemistry. There are two particularly keen students who attend all three courses. How many third-year science students are there?

Solution: Let M, P, C denote the respective sets of students attending mathematics, physics and chemistry. Once again, we are looking for the size of the union $M \cup P \cup C$. To this end, we first add $|M|$, $|P|$ and $|C|$. Students taking both mathematics and physics are double-counted, so we have to subtract $|M \cap P|$. The same applies to $|M \cap C|$ and $|P \cap C|$. However, those two students that attend all three courses are added three times and subtracted three times as well. Hence we have to add $|M \cap P \cap C|$ to make up for this. Formally, we have

$$|M \cup P \cup C| = |M| + |P| + |C| - |M \cap P| - |M \cap C| - |P \cap C| + |M \cap P \cap C|.$$

This means that there are $40 + 25 + 20 - 10 - 8 - 7 + 2 = 62$ third-year students. □

3.2 The general formula

The simple example presented in the previous section can be generalized to an arbitrary number of sets as follows:

Theorem 3.1 (Inclusion-exclusion) *Let $X_1, X_2, \ldots, X_k$ be arbitrary finite sets. For a set $I \subseteq \{1, 2, \ldots, k\}$, we denote the intersection of all sets X_i with $i \in I$ by $\bigcap_{i \in I} X_i$. Then the following formula holds:*

$$\left|\bigcup_{i=1}^{k} X_i\right| = \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ I \neq \emptyset}} (-1)^{|I|+1} \left|\bigcap_{i \in I} X_i\right|, \tag{3.1}$$

where the sum is taken over all non-empty subsets of $\{1, 2, \ldots, k\}$.

Remark: The cases $k = 2$ and $k = 3$ correspond to our two examples; for instance, the formula

$$|X_1 \cup X_2 \cup X_3| = |X_1| + |X_2| + |X_3| - |X_1 \cap X_2| - |X_1 \cap X_3| - |X_2 \cap X_3| + |X_1 \cap X_2 \cap X_3|$$

is obtained for $k = 3$.

Proof: By induction on k. For $k = 1$, the formula reduces to the trivial identity $|X_1| = |X_1|$. The induction step from k to $k + 1$ makes use of the special case $k = 2$ that was discussed in our first example:

$$\begin{aligned}
\left|\bigcup_{i=1}^{k+1} M_i\right| &= \left|\left(\bigcup_{i=1}^{k} M_i\right) \cup M_{k+1}\right| \\
&= \left|\bigcup_{i=1}^{k} M_i\right| + |M_{k+1}| - \left|\left(\bigcup_{i=1}^{k} M_i\right) \cap M_{k+1}\right| \\
&= \left|\bigcup_{i=1}^{k} M_i\right| + |M_{k+1}| - \left|\bigcup_{i=1}^{k} (M_i \cap M_{k+1})\right|
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ I\neq\emptyset}} (-1)^{|I|+1} \left|\bigcap_{i\in I} M_i\right| + |M_{k+1}| - \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ I\neq\emptyset}} (-1)^{|I|+1} \left|\bigcap_{i\in I}(M_i \cap M_{k+1})\right| \\
&= \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ I\neq\emptyset}} (-1)^{|I|+1} \left|\bigcap_{i\in I} M_i\right| + |M_{k+1}| - \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ I\neq\emptyset}} (-1)^{|I|+1} \left|\bigcap_{i\in I\cup\{k+1\}} M_i\right| \\
&= \sum_{\substack{I\subseteq\{1,\ldots,k+1\}\\ k+1\notin I, I\neq\emptyset}} (-1)^{|I|+1} \left|\bigcap_{i\in I} M_i\right| + \sum_{\substack{I\subseteq\{1,\ldots,k+1\}\\ k+1\in I}} (-1)^{|I|+1} \left|\bigcap_{i\in I} M_i\right| \\
&= \sum_{\substack{I\subseteq\{1,\ldots,k+1\}\\ I\neq\emptyset}} (-1)^{|I|+1} \left|\bigcap_{i\in I} M_i\right| .
\end{aligned}$$

This completes the induction. □

REMARK: The same formula holds true if union and intersection are interchanged (the proof being completely analogous):

$$\left|\bigcap_{i=1}^{k} X_i\right| = \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ I\neq\emptyset}} (-1)^{|I|+1} \left|\bigcup_{i\in I} X_i\right| . \tag{3.2}$$

REMARK: If we take $X_1 = X_2 = \ldots = X_n = \{x\}$ in (3.1) or (3.2), then all intersections and unions have size 1, so that we obtain another proof of the identity

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \ldots + (-1)^n \binom{n}{n} = 0,$$

see Theorem 2.3.

Quite frequently, the sets X_i are subsets of some base set, and one is interested in the number of elements that are contained in none of the X_i or not in all of the X_i. In the former case, one has to determine the cardinality of $X \setminus (X_1 \cup X_2 \ldots \cup X_k)$, which is equal to

$$|X| - \sum_{I\subseteq\{1,\ldots,k\}, I\neq\emptyset} (-1)^{|I|+1} \left|\bigcap_{i\in I} X_i\right| . \tag{3.3}$$

The second problem amounts to determining the cardinality of $X \setminus (X_1 \cap X_2 \ldots \cap X_k)$, which is given by

$$|X| - \sum_{I\subseteq\{1,\ldots,k\}, I\neq\emptyset} (-1)^{|I|+1} \left|\bigcup_{i\in I} X_i\right| . \tag{3.4}$$

Naturally, the sizes of the sets $\bigcap_{i\in I} X_i$ are not always explicitly given as in our first two examples. However, they are often easier to obtain than those that one is actually interested in. In the following section, some applications are discussed.

3.3 Applications

Problem 16 How many n-digit natural numbers are there that do not contain the digits $0, 1, 2$, but contain the digits $3, 4, 5$?

Solution: Each of the n digits is one of $3, 4, \dots, 9$. Let $M_n(Z)$ be the set of all n-digit integers that can be formed from the digits set Z. Then we are interested in the number of elements in $M_n(\{3, 4, \dots, 9\})$ that are neither in $M_n(\{4, 5, \dots, 9\})$ nor in $M_n(\{3, 5, \dots, 9\})$ nor in $M_n(\{3, 4, 6, \dots, 9\})$. Moreover, note that

$$M_n(Z_1) \cap M_n(Z_2) = M_n(Z_1 \cap Z_2),$$

i.e., numbers whose digits are all in Z_1 and also all in Z_2 have to be consist exclusively of digits in $Z_1 \cap Z_2$. Finally, we have $|M_n(Z)| = |Z|^n$ (if $0 \notin Z$; otherwise, we would have to exclude leading zeros), since there are $|Z|$ possibilities for each of the n digits. Hence we obtain from the above formula

$$\begin{aligned}
&|M_n(\{3,4,\dots,9\}) \setminus (M_n(\{4,5,\dots,9\}) \cup M_n(\{3,5,\dots,9\}) \cup M_n(\{3,4,6,\dots,9\}))| \\
&\quad = |M_n(\{3,4,\dots,9\})| - |M_n(\{4,5,\dots,9\})| - |M_n(\{3,5,\dots,9\})| - |M_n(\{3,4,6,\dots,9\}))| \\
&\quad\quad + |M_n(\{5,6,\dots,9\})| + |M_n(\{4,6,\dots,9\})| + |M_n(\{3,6,\dots,9\})| - |M_n(\{6,7,\dots,9\})| \\
&\quad = 7^n - 3 \cdot 6^n + 3 \cdot 5^n - 4^n.
\end{aligned}$$

A classical problem to which the inclusion-exclusion principle can be applied stems from number theory: *Euler's φ-function* $\varphi(n)$ is the number of integers x, $0 \leq x < n$, that are coprime to n (coprime residue classes). If $p_1, p_2, \dots, p_k$ are the prime factors of n, then a number is coprime with n if and only if it is not divisible by any of $p_1, p_2, \dots$.The number of integers $< n$ that are divisible by p_i is exactly n/p_i. Likewise, the number of integers that are divisible by all p_i with $i \in I$ (and thus by $\prod_{i \in I} p_i$) equals $n / \prod_{i \in I} p_i$. By formula (3.3), we obtain

$$\begin{aligned}
\varphi(n) &= n - \sum_{l=1}^{k} \sum_{\substack{I \subseteq \{1,\dots,k\} \\ |I|=l}} (-1)^{l-1} \frac{n}{\prod_{i \in I} p_i} \\
&= n \cdot \sum_{l=0}^{k} \sum_{\substack{I \subseteq \{1,\dots,k\} \\ |I|=l}} (-1)^{l} \prod_{i \in I} p_i^{-1} = n \cdot \prod_{i=1}^{k} (1 - \tfrac{1}{p_i}).
\end{aligned}$$

The last step follows from the fact that the term $\prod_{i \in I} \frac{1}{p_i}$ occurs in the expansion of

$$\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_k}\right)$$

with a coefficient of $(-1)^{|I|}$. Generally, the product

$$\prod_{i=1}^{k} (1 - x_i) = (1 - x_1)(1 - x_2) \dots (1 - x_k)$$

expands to

$$\sum_{I\subseteq\{1,\dots,k\}}(-1)^{|I|}\prod_{i\in I}x_i,$$

where the product over the empty set is taken to be 1, while the product

$$\prod_{i=1}^{k}(1+x_i)=(1+x_1)(1+x_2)\dots(1+x_k)$$

is simply

$$\sum_{I\subseteq\{1,\dots,k\}}\prod_{i\in I}x_i.$$

Another classical example is the number of *derangements* (*fixed-point free permutations*). One of several stories told around this problem is the following:

Problem 17 An absent-minded mathematician writes n letters and seals the envelopes before writing the addresses on the envelopes. Hence he randomly writes the addresses on the envelopes. What is the probability that none of the addresses on the letters is correct?

Solution: We have to determine the number of possible address permutations for which none of the letters is assigned the correct address. The number of all possible permutations is $n!$. Now we determine the number of permutations for which a certain set $I\subseteq\{1,2,\dots,n\}$ of letters (for simplicity's sake, we number the letters from 1 to n) is assigned the correct address.

If $m=|I|$ addresses are correct, then $n-m$ letters remain to which addresses can be assigned in $(n-m)!$ ways. Hence the number of permutations for which a set I receives correct addresses is precisely $(n-|I|)!$. By formula (3.3), we obtain the desired number

$$n!-\sum_{l=1}^{n}\sum_{\substack{I\subseteq\{1,\dots,n\}\\|I|=l}}(-1)^{l-1}(n-l)!.$$

Since there are $\binom{n}{l}=\frac{n!}{l!(n-l)!}$ subsets of cardinality l, this simplifies to

$$n!-\sum_{l=1}^{n}(-1)^{l-1}\frac{n!}{l!}=n!\sum_{l=0}^{n}\frac{(-1)^l}{l!}.$$

The probability that none of the addresses is correct, is therefore exactly

$$\sum_{l=0}^{n}\frac{(-1)^l}{l!}=1-1+\frac{1}{2}-\frac{1}{6}+\dots\pm\frac{1}{n!}.$$

For large n, this approaches the infinite sum

$$\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}=\frac{1}{e}=0.367879.$$

□

3.4 Additional problems

Problem 18 How many positive integers $1 \leq n \leq 10^{10}$ are divisible by at least one of the three primes 2,3,5?

Problem 19 How many positive integers $1 \leq n \leq 10^{10}$ satisfy $\gcd(n, 3000) = 3$?

Problem 20 How many positive integers $1 \leq n \leq 1000$ are not divisible by the third power of an integer?

Problem 21 How many positive integers $1 \leq n \leq 10^{10}$ are there that do not contain the sequence 1212 in their decimal representation?

Problem 22 How many possibilities are there to distribute 15 letters among 5 people so that each of them receives at least one letter?

Chapter 4

Enumeration by means of recursions

Many counting problems can be solved by finding appropriate recursions and solving them. For instance, we have seen that the binomial coefficients satisfy a simple recursion. In the following, we will see what strategies can be used in finding and solving a recursion.

4.1 A first example

Problem 23 We draw n straight lines in the plane, no two of which are parallel and no three of which pass through the same point. These lines divide the plane into a number of regions. Determine the number of these regions.

Solution: Let us first consider simple special cases—a principle that is often very useful. For $n = 0$, the number of regions is clearly 1, for $n = 1$ there are two regions, and for $n = 2$ we already have four. It is tempting to conjecture that the number of regions is always a power of two, but it turns out that three lines divide the plane only into seven regions.

Our next guess might be that the number of regions increases by n when the nth line is added. Indeed, this is correct, as can be seen from a more systematic approach: a newly added line intersects all the other $n-1$ lines, hence there are $n-1$ points of intersection on the new line. These divide the line into n segments, each of which divides one region into two new regions. Hence the number of regions increases by n.

If a_n denotes the number of regions that we want to determine, this means that $a_0 = 1$ and $a_n = a_{n-1} + n$. From this recursion, we also obtain an explicit formula for a_n:

$$\begin{aligned} a_n &= a_{n-1} + n = a_{n-2} + n + (n-1) = a_{n-2} + n + (n-1) + (n-2) = \dots \\ &= a_0 + n + (n-1) + (n-2) + \dots + 1 = a_0 + \frac{n(n+1)}{2} = \frac{n^2+n+2}{2}, \end{aligned}$$

making use of the well-known formula for the sum of the first n positive integers. The recursion in this example is a so-called *linear recursion*. For recursions of this type, there are standard methods to solve them, and these methods will be presented in the following sections.

Remark: It is essential to determine, in addition to a recursion, the range of its validity. This is illustrated by the following example (which is also instructive because of the method that is used):

Problem 24 A sequence a_n is given by the recursion

$$a_n = \frac{1}{\frac{1}{a_0} + \frac{1}{a_1} + \ldots + \frac{1}{a_{n-1}}} \quad (n \geq 1)$$

together with the initial value $a_0 = 1$. Determine a closed expression for a_n.

Solution: Taking the reciprocal, we obtain

$$\frac{1}{a_n} = \frac{1}{a_0} + \ldots + \frac{1}{a_{n-1}}$$

and likewise

$$\frac{1}{a_{n+1}} = \frac{1}{a_0} + \ldots + \frac{1}{a_n}.$$

Subtracting the two equations, we find

$$\frac{1}{a_{n+1}} - \frac{1}{a_n} = \frac{1}{a_n},$$

which simplifies to $a_{n+1} = \frac{a_n}{2}$. Now one could naively assume that this recursion, together with the initial value $a_0 = 1$, yields $a_n = 2^{-n}$. Indeed this already fails for $a_1 = 1$. The reason for this fact is that subtracting the two equations above is only possible for $n \geq 1$ (otherwise, the first equation is not valid), which means that $a_{n+1} = \frac{a_n}{2}$ only holds for $n \geq 1$. Indeed, one has $a_n = 2^{1-n}$, the only exception being $n = 0$. □

4.2 Finding and solving linear recursions

Problem 25 Tom gets an allowance of R100 every month, which he spends entirely on ice cream (R5), chocolate (R10) or cookies (R10). Every day, he buys exactly one of these until he runs out of money. In how many possible ways can Tom spend his money? His older brother Phil gets an allowance of R150, which he also spends on ice cream, chocolate and cookies only. How many possible ways does he have?

Solution: We study a more general problem: if the allowance is $5n$ (n any positive integer; note that the prices are all divisible by 5), how many ways are there? Let this number be denoted by a_n; then it is easy to see that $a_1 = 1$ and $a_2 = 3$. Next we deduce a recursion for a_n: the first thing that Tom buys can be either ice cream (so that he is left with $5(n-1)$ and has a_{n-1} possibilities for the rest) or chocolate or cookies (in each of these cases, the remaining amount is $5(n-2)$, so that he is left with a_{n-2} possibilities). This shows that

$$a_n = a_{n-1} + 2a_{n-2}$$

holds. Note that this even remains true if we set $a_0 = 1$ (without money, there is only one option, which is to buy nothing at all). One obtains the following sequence:

n	1	2	3	4	5	6	7	8	9	10
a_n	1	3	5	11	21	43	85	171	341	683

An explicit formula is given by

$$a_n = \frac{1}{3}\left(2^{n+1} + (-1)^n\right),$$

which can be proven by means of induction. A method to determine such an explicit formula from a recursion will be discussed in the following. Plugging in $n = 20$ and $n = 30$ respectively, we find that Tom has 699051 options, while Phil has 715827883 different ways to spend his money.

How does one find the solution to such a linear recursion? There is a general approach that can be used to solve linear recursions, i.e., recursions of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_d a_{n-d} \tag{4.1}$$

with certain constant coefficients $c_1, c_2, \ldots, c_d$. Simple examples like $a_n = 2a_{n-1}$ (initial value $a_0 = 1$), whose explicit solution $a_n = 2^n$ can be guessed easily, suggest that one should look for solutions of the form $a_n = q^n$. This approach leads to the condition

$$q^n = c_1 q^{n-1} + c_2 q^{n-2} + \ldots + c_d q^{n-d}$$

or (multiplying by q^{d-n})

$$q^d = c_1 q^{d-1} + c_2 q^{d-2} + \ldots + c_d.$$

This means that q has to be a solution of the so-called *characteristic equation*

$$q^d - c_1 q^{d-1} - c_2 q^{d-2} - \ldots - c_d = 0,$$

which is $q^2 - q - 2 = 0$ in our specific case. The solutions are $q_1 = -1$ and $q_2 = 2$, and indeed the two sequences $a_n = (-1)^n$ and $a_n = 2^n$ are solutions to our recursion. It remains to take the initial values into account. We notice that arbitrary multiples of a solution are again solutions, and that the sum of two solutions is also a solution. In this case, every sequence of the form

$$a_n = A \cdot (-1)^n + B \cdot 2^n$$

is a solution to our recursion, so it only remains to determine appropriate values for A and B in agreement with our initial values. To this end, we solve the system of equations

$$a_0 = A + B = 1 \text{ and } a_1 = -A + 2B = 1,$$

which yields $A = \frac{1}{3}$ and $B = \frac{2}{3}$. Finally, this means that the solution to our problem is given by

$$a_n = \frac{1}{3}\left(2^{n+1} + (-1)^n\right),$$

as it was claimed before. □

Without getting too far into the details, let us mention that the solution to any linear recursion of the type (4.1) has the form

$$\begin{aligned}a_n = {} & C(1,1)q_1^n + C(1,2)nq_1^n + \ldots + C(1,l(1))n^{l(1)-1}q_1^n + C(2,1)q_1^n + C(2,2)nq_2^n + \\ & \ldots + C(2,l(2))n^{l(2)-1}q_2^n + \ldots + C(r,l(r))n^{l(r)-1}q_r^n,\end{aligned}$$

where $q_1, q_2, \ldots, q_r$ are the zeros of the characteristic polynomial and $l(i)$ denotes the *multiplicity* of q_i as a zero, i.e., the highest power l such that $(q - q_i)^l$ is a factor of the characteristic polynomial. In our example above, -1 and 2 were simple zeros, so that the solution had to be of the form $a_n = A \cdot (-1)^n + B \cdot 2^n$.

Linear recursions become slightly more complicated if they are *non-homogeneous*, i.e., if there are additional terms on the right hand side of the equation that depend on n (but not on the sequence a_n), as in the recursion

$$a_n = a_{n-1} + n.$$

Such an additional term (usually a polynomial in n or an exponential function λ^n or a mixture of the two) is called a *non-homogeneous term*.

Generally, the solution to such an equation is obtained as the sum of a particular solution and the general solution to the corresponding homogeneous equation. In this case, the homogeneous equation is

$$a_n = a_{n-1}$$

with the obvious general solution $a_n = C$, i.e., a constant sequence. To obtain a particular solution to the non-homogeneous recursion, where the non-homogeneous term is of the form

$$P(n)\lambda^n$$

for a polynomial P, one uses the so-called *ansatz* $a_n^* = Q(n)n^{l(\lambda)}\lambda^n$, where $Q(n)$ is a polynomial (to be determined) of the same degree as P, and $l(\lambda)$ denotes the multiplicity of λ as a zero of the characteristic polynomial (0, if it is not a zero at all). If the non-homogeneous term is a sum of such expressions, one takes the sum of the corresponding a_n^*.

In our specific example, we try $a_n^* = (An + B)n$ with undetermined coefficients A and B, since $\lambda = 1$ is a single zero of the characteristic polynomial $q-1$. Comparing coefficients in the recursion, we find A and B: from

$$(An + B)n = (A(n-1) + B)(n-1) + n,$$

we get $B = B - 2A + 1$ and $0 = A - B$, so $A = B = \frac{1}{2}$. Hence the general solution to our recursion is

$$\frac{n^2 + n}{2} + C,$$

and from the initial value $a_0 = 1$, we find the formula for the number of regions that we obtain by drawing n lines in the plane:

$$a_n = \frac{n^2 + n}{2} + 1.$$

As a last example of an application of this technique, let us consider the recursion

$$a_n = 3a_{n-1} - 2a_{n-2} + n2^n \quad (n \geq 2)$$

with initial values $a_0 = a_1 = 1$. The characteristic polynomial of the associated homogeneous equation

$$a_n = 3a_{n-1} - 2a_{n-2}$$

is given by $q^2 - 3q + 2$ with the two solutions $q_1 = 1$ and $q_2 = 2$. Hence the general solution to the homogeneous equation is $C_1 + C_2 2^n$. In order to find a particular solution to the non-homogeneous solution, we put $a_n^* = (An + B)n2^n$ (note that 2 is already a simple zero of the characteristic polynomial!):

$$(An + B)n2^n = 3(A(n-1) + B)(n-1)2^{n-1} - 2(A(n-2) + B)(n-2)2^{n-2} + n2^n.$$

We divide by 2^{n-1} and expand:

$$2An^2 + 2Bn = 2An^2 + 2(B - A + 1)n - (A + B).$$

Comparing coefficients, we obtain $B = B - A + 1$ and $A + B = 0$, which yields $A = 1$, $B = -1$. Hence the general solution to our recursion is given by

$$a_n = (n-1)n2^n + C_1 + C_2 2^n.$$

We plug in the initial values to determine $C_1 = 1$ and $C_2 = 0$, so that we finally end up with

$$a_n = (n-1)n2^n + 1.$$

□

The following problem leads to the well-known *Fibonacci numbers*:

Problem 26 All the houses on one side of a certain street are to be painted either yellow or red. In how many ways can this be done if there are n houses and there may not be two red houses next to each other?

Solution: Let a_n be the number that we want to determine. We distinguish two cases:

- If the first house is painted yellow, then the remaining houses can be painted in any of the feasible a_{n-1} ways, the first house can be neglected.
- If the first house is painted red, then the second house must be painted yellow. Using the same argument as before, we see that there are a_{n-2} possibilities for the remaining houses.

Hence we have the recursion

$$a_n = a_{n-1} + a_{n-2}$$

with initial values $a_1 = 2$ and $a_2 = 3$. This yields the sequence $2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$ of so-called *Fibonacci numbers*. Usually, this sequence is defined by the initial values $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$. It is easy to verify that $f_n = a_{n-2}$ holds for $n \geq 3$. Applying the same steps as in the previous example (which is a good exercise), one obtains *Binet's formula*:

$$f_n = a_{n-2} = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

4.3 Generating functions

Another possibility to solve linear recursion is the method of *generating functions*. This approach is also useful in solving more complicated recursions. The main idea is to associate an infinite series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

with a sequence a_n. Let us first consider the generating functions associated with some simple sequences:

1. The constant sequence $a_n = a$ corresponds to the generating function

   $$A(x) = \sum_{n=0}^{\infty} a x^n = a \sum_{n=0}^{\infty} x^n = \frac{a}{1-x},$$

   as can be seen from the sum formula for the geometric series.

2. A slightly more general example is the geometric sequence $a_n = ab^n$, whose generating function is found to be

   $$A(x) = \sum_{n=0}^{\infty} ab^n x^n = a \sum_{n=0}^{\infty} (bx)^n = \frac{a}{1-bx}.$$

3. If we set $a_n = \binom{N}{n}$ and note that the equation $\binom{N}{n} = 0$ holds for $n > N$, then the binomial theorem shows that

   $$A(x) = \sum_{n=0}^{\infty} \binom{N}{n} x^n = \sum_{n=0}^{N} \binom{N}{n} x^n = (1+x)^N.$$

   Let us mention without proof that this identity remains true for non-integer values of N, i.e.,

   $$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = (1+x)^\alpha$$

for arbitrary real α (but only for $-1 < x < 1$; for other real values of x, the series does not converge). In particular, the following special cases deserve to be mentioned:

- $\alpha = -1$: here we have $\binom{-1}{n} = \frac{(-1)(-2)\dots(-n)}{(1)(2)\dots(n)} = (-1)^n$ and thus

$$\sum_{n=0}^{\infty} (-1)^n x^n = (1+x)^{-1}.$$

  The formula for the geometric series is thus a special case.

- $\alpha = -2$: here we have $\binom{-2}{n} = \frac{(-2)(-3)\dots(-n-1)}{(1)(2)\dots(n)} = (-1)^n(n+1)$ (generally, we have $\binom{-m}{n} = (-1)^n\binom{n+m-1}{m-1}$) and thus

$$\sum_{n=0}^{\infty} (-1)^n (n+1) x^n = (1+x)^{-2}.$$

  Replacing x by $-x$, one obtains

$$1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}.$$

- $\alpha = \frac{1}{2}$: we have

$$\begin{aligned}
\binom{1/2}{n} &= \frac{(1/2)(-1/2)(-3/2)\dots(1/2-n+1)}{n!} = \frac{(-1)^{n-1}\cdot 1\cdot 3\cdot \ldots \cdot (2n-3)}{2^n \cdot n!} \\
&= \frac{(-1)^{n-1}\cdot 1\cdot 2\cdot 3 \ldots \cdot (2n-2)}{2^n\cdot n!\cdot 2\cdot 4\cdot \ldots \cdot (2n-2)} = \frac{(-1)^{n-1}(2n-2)!}{2^n\cdot n!\cdot 2^{n-1}\cdot (n-1)!} \\
&= \frac{(-1)^{n-1}}{2^{2n-1}n}\cdot \binom{2n-2}{n-1}
\end{aligned}$$

  for $n \geq 1$ ($\binom{1/2}{0} = 1$), so that we obtain

$$\sqrt{1+x} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{2n-1}n}\cdot \binom{2n-2}{n-1} x^n,$$

  a fact that we are going to use in the last example of this section.

Moreover, we need the following elementary rules for generating functions:

1. If $A(x)$ is the generating function of a_n, then $cA(x)$ is the generating function of ca_n.
2. If $A(x)$ and $B(x)$ are the generating functions of a_n and b_n respectively, then $A(x) + B(x)$ is the generating function of $a_n + b_n$.

3. If $A(x)$ is the generating function of a_n, then $A(qx)$ is the generating function of $q^n a_n$.

4. If $A(x)$ and $B(x)$ are the generating functions of a_n and b_n respectively, then $A(x)B(x)$ is the generating function of $c_n := \sum_{k=0}^{n} a_k b_{n-k}$.

The proof of the first three rules is left as an exercise for the interested reader (the rules are essentially the same as for polynomials). Let us have a closer look at the last of our rules (the so-called "Cauchy product"): we multiply the product out (cf. the remarks on the proof of Theorem 2.5) to obtain

$$\begin{aligned}\left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{m=0}^{\infty} b_m x^m\right) &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} a_k b_m x^{k+m} \\ &= \sum_{n=0}^{\infty}\sum_{k+m=n} a_k b_m x^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n,\end{aligned}$$

which proves the fourth rule. One can now use generating functions to solve recursions by first determining $A(x)$ by some algebraic manipulations and then obtaining the sequence a_n from $A(x)$. This is by far not the only application of generating functions, but we will restrict ourselves to this application here. First of all, let us consider a linear recursion that we have already solved by means of a different method. After that, we will deal with some examples of non-linear recursions that are solved by means of generating functions. It is worth mentioning that the proofs of some of the identities for binomial coefficients that were given in Section 2 are essentially generating function proofs as well.

Let us consider our standard example of a linear recursion again:

$$a_n = 3a_{n-1} - 2a_{n-2} + n2^n \quad (n \geq 2)$$

with initial values $a_0 = a_1 = 1$. Let the generating function of the sequence a_n be $A(x)$. We multiply the equation by x^n and sum over all $n \geq 2$ (since the recursion only holds within this range!). Then it follows that

$$\sum_{n=2}^{\infty} a_n x^n = 3\sum_{n=2}^{\infty} a_{n-1}x^n - 2\sum_{n=2}^{\infty} a_{n-2}x^n + \sum_{n=2}^{\infty} n2^n x^n.$$

Now let us consider the terms of this identity separately. The first sum is $A(x)$ without its first two terms, hence

$$\sum_{n=2}^{\infty} a_n x^n = A(x) - a_0 - a_1 x = A(x) - 1 - x.$$

In the second sum we substitute m for $n-1$ and obtain

$$\sum_{n=2}^{\infty} a_{n-1}x^n = \sum_{m=1}^{\infty} a_m x^{m+1} = x\sum_{m=1}^{\infty} a_m x^m = x(A(x)-a_0) = x(A(x)-1).$$

Likewise, we have

$$\sum_{n=2}^{\infty} a_{n-2}x^n = \sum_{m=0}^{\infty} a_m x^{m+2} = x^2 A(x).$$

Finally, we have to determine the sum $\sum_{n=2}^{\infty} n2^n x^n$. To this end, we apply the formula

$$\sum_{n=0}^{\infty}(n+1)x^n = \frac{1}{(1-x)^2}$$

that we have seen before. Then we obtain

$$\sum_{n=0}^{\infty} nx^n = \sum_{n=0}^{\infty}(n+1)x^n - \sum_{n=0}^{\infty} x^n = \frac{1}{(1-x)^2} - \frac{1}{1-x} = \frac{x}{(1-x)^2}.$$

Hence,

$$\sum_{n=2}^{\infty} n2^n x^n = \sum_{n=0}^{\infty} n(2x)^n - 2x = \frac{2x}{(1-2x)^2} - 2x.$$

So we have the identity

$$A(x) - 1 - x = 3x(A(x)-1) - 2x^2A(x) + \frac{2x}{(1-2x)^2} - 2x,$$

which we solve for $A(x)$. The result is the identity

$$A(x) = \frac{1-6x+20x^2-16x^3}{(1-x)(1-2x)^3}.$$

In order to deduce a formula for a_n, we decompose $A(x)$ into *partial fractions*. Once one has factored the denominator into factors of the form $1+ax$, one can rewrite the fraction as a sum of simpler fractions which has the following form in this specific example:

$$\frac{1-6x+20x^2-16x^3}{(1-x)(1-2x)^3} = \frac{A}{1-x} + \frac{B_1}{1-2x} + \frac{B_1}{(1-2x)^2} + \frac{B_3}{(1-2x)^3}.$$

Generally, one can write a rational function as a sum of terms of the form $\frac{C}{(1+ax)^k}$, where the exponent k ranges over all positive integers up to the multiplicity of $1+ax$ as a factor of the original denominator. Now one can determine the precise decomposition by comparing coefficients (this is always possible, but we skip the proof of this fact here). Another possibility is the following: we multiply both sides of the equation by $1-x$ and plug in $x=1$. Then it follows immediately that $A=1$, since almost all terms vanish. Likewise, one could also multiply by $(1-2x)^3$ and plug in $x = \frac{1}{2}$ (thus obtaining B_3), then subtract the last summand on both sides (and determine B_2), etc. Here, however, we see directly that

$$\frac{1-6x+20x^2-16x^3}{(1-x)(1-2x)^3} - \frac{1}{1-x} = \frac{8x^2}{(1-2x)^3}$$

holds. Hence we have the following expression for $A(x)$:

$$A(x) = \frac{1}{1-x} + \frac{8x^2}{(1-2x)^3}.$$

By means of the binomial series, we can now determine a_n:

$$\begin{aligned}
A(x) &= \frac{1}{1-x} + \frac{8x^2}{(1-2x)^3} \\
&= \sum_{n=0}^{\infty} x^n + 8x^2 \sum_{n=0}^{\infty} \binom{-3}{n} (-2x)^n \\
&= \sum_{n=0}^{\infty} x^n + 8x^2 \sum_{n=0}^{\infty} (-1)^n \binom{n+2}{2} (-2x)^n \\
&= \sum_{n=0}^{\infty} x^n + 4x^2 \sum_{n=0}^{\infty} (n+2)(n+1)(2x)^n \\
&= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (n+2)(n+1)(2x)^{n+2} \\
&= \sum_{n=0}^{\infty} x^n + \sum_{n=2}^{\infty} n(n-1)(2x)^n \\
&= \sum_{n=0}^{\infty} \left(1 + n(n-1)2^n\right) x^n.
\end{aligned}$$

Hence we have $a_n = 1 + n(n-1)2^n$, which agrees with our first solution. □

However, the main strength of generating functions lies in the fact that non-linear recursions can be treated as well, as in the following examples:

Problem 27 We want to form a configuration of cans according to the following rules: each can is supported by precisely two cans below it, and none of the rows has a gap (see Figure 4.1). How many different configurations are there under this condition, if there are n cans in the first row?

Solution: As usual, let a_n be the number we want to determine. A possible configuration consists of any feasible configuration with at most $n-1$ cans in the bottom row that is placed on top of a row of n cans. If there are i cans in the second row, then there are a_i

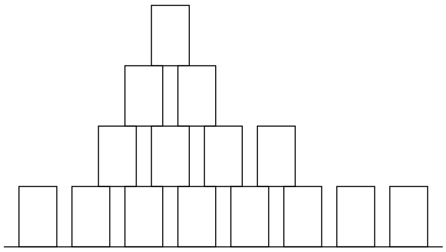

Figure 4.1: A possible configuration of cans.

possibilities for the configuration and $n-i$ possibilities for the position. The only exception is $i = 0$, when there is no choice of position. Hence we have the recursion

$$a_n = 1 + \sum_{i=1}^{n-1} (n-i)a_i = 1 + \sum_{i=0}^{n} (n-i)a_i \quad (n \geq 1)$$

with initial value $a_0 = 0$ (in this case, we define it in such a way that the calculations become simpler). If $A(x)$ is the generating function, then we obtain from this recursion by means of multiplication by x^n and summation over all $n \geq 1$ according to the rule for the product of two generating functions

$$A(x) = \sum_{n=1}^{\infty} x^n + \left(\sum_{n=0}^{\infty} nx^n \right) A(x) = \frac{x}{1-x} + \frac{xA(x)}{(1-x)^2},$$

which simplifies to $A(x) = \frac{x-x^2}{1-3x+x^2}$. Applying a partial fraction decomposition as in the previous example, we can deduce an explicit formula for the elements of the sequence. It turns out that these elements are exactly the Fibonacci numbers with odd index, i.e., $1, 2, 5, 13, \ldots$. We would like to prove this directly by means of generating functions. To this end, we need the generating function for the Fibonacci numbers, whose recursion is given by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$. We set $F(x) = \sum_{n=0}^{\infty} f_n x^n$ and obtain

$$\sum_{n=2}^{\infty} f_n x^n = \sum_{n=2}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n$$

or

$$F(x) - x = xF(x) + x^2 F(x),$$

which yields $F(x) = \frac{x}{1-x-x^2}$. We are interested in the odd elements of the sequence only. These can be obtained as follows: note that $\frac{1}{2}(1-(-1)^n)$ equals 1 if n is odd and 0 otherwise. Hence it follows that

$$\sum_{\substack{n=0 \\ n \text{ odd}}}^{\infty} f_n x^n = \frac{1}{2} \sum_{n=0}^{\infty} (1-(-1)^n) f_n x^n = \frac{1}{2} \left(F(x) - F(-x) \right).$$

In our case, this is equal to $\frac{x(1-x^2)}{1-3x^2+x^4}$. Since this agrees with

$$A(x^2)/x = \sum_{n=1}^{\infty} a_n x^{2n-1},$$

the elements of our sequence a_n are indeed the Fibonacci numbers with odd index. □

As a last application of generating functions, we deduce the formula for the Catalan numbers in another way. For this purpose, we first determine a recursion for the number a_n of "up-down-sequences" that stay above the x-axis and consist of n up- and n down-steps. Let us consider the first time when we reach the x-axis again. This happens after $2k$ steps (k up, k down) for some k with $1 \leq k \leq n$. The first of these $2k$ steps has to be an up-step, the last one a down-step (see Figure 4.2). The sequence between forms another feasible up-down-sequence, since is stays above the line $y = 1$ by our definition of k. Hence there are a_{k-1} possibilities, and a_{n-k} possibilities for the remaining $2(n-k)$ steps. So we have the recursion

$$a_n = \sum_{k=1}^{n} a_{k-1} a_{n-k} \quad (n \geq 1).$$

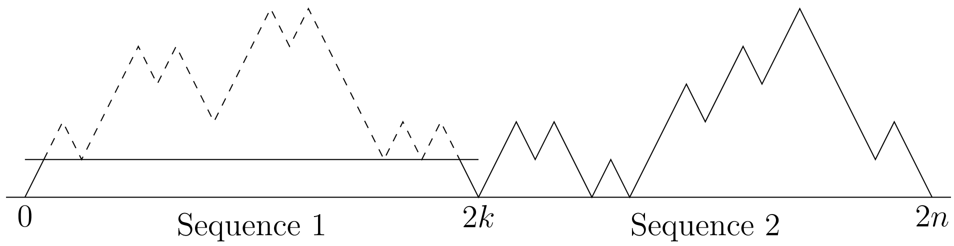

Figure 4.2: Proof of the recursion for the Catalan numbers.

Together with the initial value $a_0 = 1$, this uniquely defines the sequence. As before, we first multiply by x^n and sum over $n \geq 1$. For the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$, it follows that

$$\begin{aligned} A(x) - 1 &= \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_{k-1} a_{n-k} x^n \\ &= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} a_k a_{n-k-1} x^n = x \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} a_k a_{n-1-k} x^{n-1} \\ &= x \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k a_{n-k} x^n = x A(x)^2. \end{aligned}$$

The solutions to this quadratic equation are given by $A(x) = \frac{1\pm\sqrt{1-4x}}{2x}$. Since $xA(x)$ has to be 0 if $x = 0$ is plugged in, we have to choose the negative sign. Hence we have

$$A(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Now we plug in the known series for the function $\sqrt{1+x}$:

$$\begin{aligned} A(x) &= \frac{1}{2x}\left(1 - 1 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{2n-1}n} \cdot \binom{2n-2}{n-1}(-4x)^n\right) \\ &= \frac{1}{2x}\sum_{n=1}^{\infty} \frac{2}{n} \cdot \binom{2n-2}{n-1} x^n \\ &= \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} x^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n. \end{aligned}$$

It follows directly that $a_n = \binom{1}{n+1}\binom{2n}{n}$, which gives us an alternative way to derive the formula for the number of "change sequences" in Problem 8.

4.4 Additional problems

Problem 28 In how many ways can a $2 \times n$-rectangle be filled with 1×2-dominoes and 2×2-squares?

Problem 29 In how many ways can a $2 \times 2 \times n$-block be filled with $1 \times 1 \times 2$-tiles? Two configurations are counted as different if at least one tile is placed differently, even if the configurations can be obtained from each other by rotations.

Problem 30 (IMO 1979) Let A and E be two opposite vertices of a regular octagon ABCDEFGH. A frog starts at vertex A and jumps along one of the edges at every move. Once the frog reaches E, it stays there. Determine the number of possible paths from A to E of length n (i.e., sequences of points, starting at A and ending at E, such that all points except for the last one are different from E and any two consecutive points of the sequence are adjacent). For $n = 4$, for instance, there are the two paths A-B-C-D-E and A-H-G-F-E. A possible path for $n = 8$ would be A-B-A-H-G-F-G-F-E.

Problem 31 A coin is tossed until we obtain a sequence of an odd number of "heads", followed by one "tail", for the first time. How many possible sequences of n tosses are there? (One possible sequence of length 13 would be HHHHTTHHTHHHT.)

Problem 32 In how many ways can the vertices of a regular n-gon be coloured in such a way that no two adjacent vertices are coloured with the same colour?

Problem 33 (Ireland 1997) How many 1000-digit numbers are there, such that

- all digits are odd, and
- the difference between any two subsequent digits is exactly 2?

Problem 34 (Hungary/Israel 1997) How many sequences of length 1997 formed by the letters A,B,C are there, for which the number of A's and the number of C's are both odd?

Problem 35 (Mongolia 2001) A sequence of n points is given on a line. How many possibilities are there to colour them with two colours (red and blue) in such a way that for any subsequence of consecutive points the number of red points in this subsequence differs from the number of blue points by at most 2?

Problem 36 (Great Britain 2001) Twelve people sit around a round table. In how many ways can they shake hands in six pairs, if no two of the pairs may cross?

Problem 37 (Czech/Slovak competition 2004) For a positive integer n, let p_n be the number of words of length n using only the letters A and B which do not contain AAAA or BBB as a subword. Determine

$$\frac{p_{2004} - p_{2002} - p_{1999}}{p_{2000} + p_{2001}}.$$

Chapter 5

The pigeonhole principle

5.1 The principle and a first example

The main idea of the pigeonhole principle is very simple, yet it can be applied to a variety of problems. It is based on the following fact:

Theorem 5.1 *If $n+1$ objects are put into n pigeonholes, then there is at least one pigeonhole that contains at least two of the objects. More generally: if more than kn objects are put into n pigeonholes, then at least one of them has to contain at least $k+1$ of the objects.*

Proof: Suppose that each of the pigeonholes contains k objects or less; then the total number of objects is less than kn, a contradiction. $\square$

There are many simple examples that illustrate how the pigeonhole principle is applied, for instance:

Problem 38 Prove that there are 100 000 South Africans that have the same birthday.

Solution: The “pigeonholes” are the 366 different birth dates in this case, and the “objects are the $> 40\,000\,000$ South Africans. Since the inequality $40\,000\,000 > 99\,999 \cdot 366$ holds, the statement follows immediately. $\square$

5.2 Various applications

The main difficulty in the application of the pigeonhole principle—which often requires some ingenuity—lies in defining the right “objects” and “pigeonholes”. Let us first consider an example from number theory:

Problem 39 Can one find four resp. five distinct positive integers with the property that the sum of any three is a prime number?

Solution: Four such numbers can be found: $1, 3, 7$ und 9 satisfy the required condition. However, there is no set of five numbers with this property: to prove this fact, consider the numbers modulo 3. Then we have three "pigeonholes", namely the residue classes 0, 1 and 2. If there is a number in each of these three classes, then their sum is divisible by 3, and since the numbers are distinct, their sum is > 3. So it cannot be a prime number.

Otherwise, the numbers are only distributed among two different residue classes, which shows that one of the residue classes must contain at least three of the numbers. But this again implies that their sum is divisible by 3, so that we can apply the same argument as before. □

The following example is already more complex:

Problem 40 Prove that any sequence of $nm + 1$ distinct real numbers either contains an increasing subsequence of length $n + 1$ or a decreasing subsequence of length $m + 1$.

Proof: Let $x_1, x_2, \ldots$ be the sequence, and let $\ell(i)$ be the length of the longest increasing subsequence whose first element is x_i. If there is an i with $\ell(i) \geq n+1$, then there is nothing to prove. Otherwise we have $1 \leq \ell(i) \leq n$ for all i, which means that there are only n possibilities (pigeonholes!) for the values of $\ell(i)$. By the pigeonhole principle, there must be a certain value L such that there are at least $m + 1$ distinct elements of the sequence with $\ell(i) = L$. Now we prove that these $m + 1$ elements must form a decreasing sequence. If not, then we can find two elements x_i, x_j with $i < j$, $x_i < x_j$ and $\ell(i) = \ell(j) = L$. Take the longest increasing subsequence that starts with x_j, and add x_i to the sequence. The result is an increasing subsequence of length $L+1$, which contradicts our assumption that $\ell(i) = L$. This proves our claim that the elements x_i with $\ell(i) = L$ form a decreasing sequence of length $m + 1$ (or more). □

Quite frequently, the pigeonhole principle is embedded in geometry problems, such as the following:

Problem 41 Prove: among any five points inside an equilateral triangle of side length 1, there are two points whose distance is at most $\frac{1}{2}$.

Solution: We divide the triangle into four congruent parts, as shown in Figure 5.1. Then there must be two points out of the five that belong to the same part. The distance between those two points is clearly at most the side length of the smaller triangles, which is $\frac{1}{2}$. □

The pigeonhole principle can also often be applied in a number-theoretic context. We have already seen one such example, the following problem shows another application in number theory:

Problem 42 (Austria 2005) Prove: there exist infinitely many multiples of 2005 that contain each of the ten digits with the same frequency.

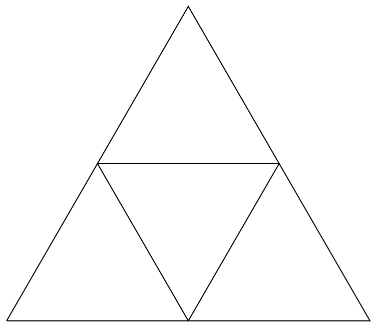

Figure 5.1: Dividing the triangle.

Solution: Several approaches are possible, and one of them makes use of the pigeonhole principle. Consider numbers of the form

$$12345678901234567890\ldots1234567890.$$

Clearly all such numbers are divisible by 5 and contain an equal number of 0s, 1s, etc. Since there are infinitely many such numbers, there must be two that leave the same remainder upon division by 401. Subtracting these two, we obtain a number of the form

$$12345678901234567890\ldots1234567890000\ldots000$$

that is divisible by 401. Since 401 and 10 are coprime, this implies that there must also be a number of the form

$$12345678901234567890\ldots1234567890$$

that is divisible by 401. This number is then also divisible by 2005 and satisfies the required condition. □

Let us conclude this chapter with a famous application of the pigeonhole principle:

Problem 43

1. Prove that among any six people there are always either three who know each other or three who are unknown to each other.

2. 17 scientists regularly discuss three particular topics. Any two of them only discuss one of the three topics with each other. Prove that there is a group of three scientists who only discuss one specific topic amongst each other.

Solution:

1. Consider an arbitrary person X. Of the five people remaining, there are either three that are known to X, or three that X does not know yet. The two cases are analogous, so let us assume that the former case is true, and that X knows Y_1, Y_2 and Y_3. It remains to distinguish two simple cases:

- Two of Y_1, Y_2, Y_3 know each other. Then X and these two form a triple of three people who mutually know each other.
- Otherwise, Y_1, Y_2, Y_3 form a triple of three people who mutually don't know each other.

2. Consider one specific scientist W. Among the 16 others, there are at least six who discuss the same topic, say topic A, with W. If there are two among these six who also discuss topic A with each other, then we are already done. Otherwise, these six are restricted to only two topics (topic B and topic C, say). If we identify "topic B" with "know" and "topic C" with "do not know", then it becomes clear that we have reduced the problem to the first part. □

This example can be generalised further—we will return to it in Section 7.4.

5.3 Additional problems

Problem 44 Let $p > 3$ be a prime, and let n be a positive integer. Prove: if the number p^n has exactly 100 digits, then one digit occurs at least 11 times.

Problem 45 An infinite sequence of digits $(0 - 9)$ is given. Prove that one can always find a block of digits inside this sequence such that the number represented by this block is divisible by 37. For instance, the sequence

$$12784389674598694\ldots$$

contains the number $74 = 2 \cdot 37$.

Problem 46 (Iceland 1996) Prove: among 52 pairwise distinct integers, there are always two distinct integers such that 100 divides their sum or their difference.

Problem 47

1. Let five distinct points in the plane with integer coordinates be given. We draw all five line segments connecting them. Prove that one of these line segments contains a point with integer coordinates in its interior.

2. The vertices of a convex pentagon have integer coordinates. Prove that the area of this pentagon is at least $5/2$.

Problem 48 (Germany 2004) A set of n disks $K_1, \ldots, K_n$ with radius r is given in the plane. We know that no point of the plane is covered by more than 2003 of these disks. Prove that no disk K_i intersects more than 14020 other disks (the boundary counts as part of a disk).

Problem 49 "Math lotto" is played as follows: a player marks six squares on a 6×6-square. Then six "losing squares" are drawn. A player wins if none of the losing squares are marked on his lottery ticket.

- Prove that one can complete nine lottery tickets in such a way that at least one of them wins.
- Prove that this is not possible with only eight tickets.

Problem 50 The primary school of a remote village is attended by 20 children. Any two of them have a common grandfather. Prove that there are 14 children among these 20 that have a common grandfather.

Problem 51 (IMO 1978) An international society has members from six different nations. The list of these members comprises 1978 names, numbered from 1 to 1978. Prove that there is at least one member whose number is the sum of the numbers of two members from the same country or twice the number of a member from the same country.

Problem 52 41 rooks are placed on a 10×10-chessboard. Prove that there are five rooks among these 41 that do not attack each other (i.e., no two of them are in the same horizontal or vertical line).

Problem 53 (Czech/Slovak contest 1998) Let a set of 14 pairwise distinct positive integers be given. Prove that one can find two subsets $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$ of this set with the same number of elements and empty intersection such that the sums

$$x = \frac{1}{a_1} + \frac{1}{a_2} + \dots \frac{1}{a_k}$$

and

$$y = \frac{1}{b_1} + \frac{1}{b_2} + \dots \frac{1}{b_k}$$

differ by less than 0.001, i.e., $|x - y| < 0.001$.

Problem 54 (Canada 2003) A set of n points is given in the plane such that the distance between any two of them is greater than 1. Prove that one can choose at least $n/7$ of these points such that the distance between any two of them is greater than $\sqrt{3}$.

Chapter 6

Potential functions and invariants

Invariants and potential functions belong to those combinatorial techniques that are hard to treat theoretically. There is also not always a clear distinction between these two in the literature. Here, we use the two notions in the following sense:

- An *invariant* is a parameter that remains constant under certain operations or is independent of certain choices.
- A *potential function* is typically a value which can be shown to be increasing/decreasing/nonincreasing/nondecreasing under certain operations.

6.1 Invariants

It is easiest to explain these concepts by means of examples. Let us begin with the following classical problem:

Problem 55 Two diagonally opposite corners of an 8×8-chessboard are removed. Is it possible to completely cover the remaining 62 squares with 31 2×1-dominoes?

Solution: Such a tessellation of the chessboard is impossible. To see why, colour the squares in the usual way (black/white), and note that a domino always covers precisely one white square and one black square, regardless of its position. Hence the number of white squares that are covered has to be the same as the number of black squares that are covered—we have found our first invariant. Since our chessboard (of which two diagonally opposite squares have been removed) consists of 32 squares of one colour and 30 squares of the other colour, the task cannot be completed. □

Invariants can be much more complicated than in our first example, but still use the same ideas. As an example, consider the following problem:

Problem 56 (IMO 2004) Determine all $m \times n$ rectangles that can be covered with "hooks" made up of 6 unit squares, as in the figure: Rotations and reflections of hooks

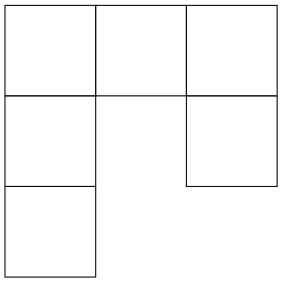

Figure 6.1: The "hook".

are allowed. The rectangle must be covered without gaps and overlaps. No part of a hook may cover area outside the rectangle.

Solution: If such a hook is placed on the rectangle, a hole remains in the middle that has to be filled by another hook. Since there is only one empty adjacent square, the hole has to be covered by one of the ends of the second hook. Hence there are only two possibilities to achieve this (Figure 6.2).

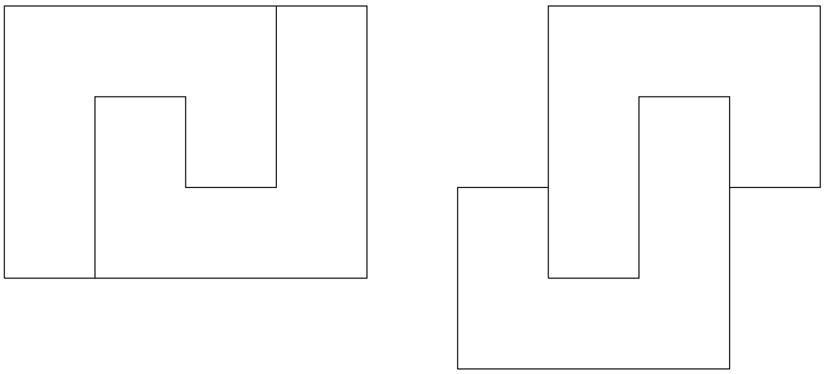

Figure 6.2: The two possibilities to join two hooks.

Hence one has two different shapes that can be used to cover the rectangle: a 3×4-rectangle and a "skew" rectangle. Both have an area of 12 square units, hence the area of the rectangle has to be a multiple of 12. We distinguish two cases:

One of the side lengths is divisible by 4. Since one of the side lengths has to be divisible by 3, the rectangle has to be of the form $3k \times 4l$ or $k \times 12l$. A rectangle of the form $3k \times 4l$ can be covered trivially by 3×4-rectangles. Hence it is possible to cover a $3 \times 12l$ or a $4 \times 12l$-rectangle, and as a consequence any rectangle of the form $(3a + 4b) \times 12l$, by cutting it into stripes.

Every positive integer $\neq 1, 2, 5$ can be written as $3a + 4b$, as can be seen by a simple induction. A $1 \times 12l$- or a $2 \times 12l$-rectangle can clearly not be covered, since there is not enough space for a hook. Let us finally consider the $5 \times 12l$-rectangle. As can be seen from Figure 6.3, the square that is marked by an X has to be covered along with the lower left corner. Since the same holds true analogously for the upper left corner, there must be overlapping pieces, which shows that the $5 \times 12l$-rectangle cannot be covered. This completes the first case.

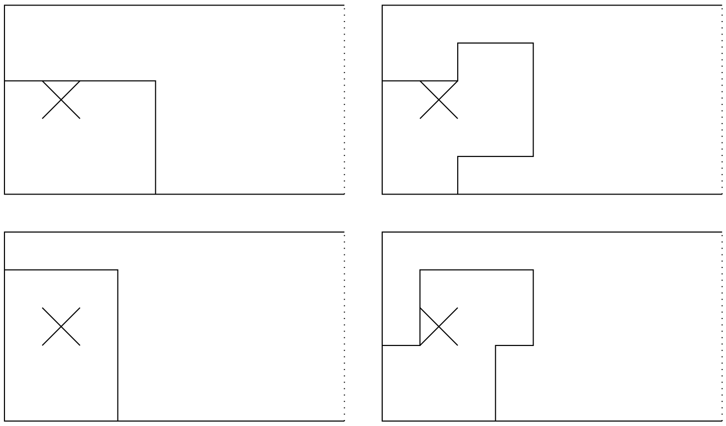

Figure 6.3: Possibilities to cover the corner.

Neither of the two side lengths is divisible by 4. In this case, both side lengths have to be even. Now colour the horizontal rows of the rectangle alternatingly black and white. Since the side lengths are even, the number of black squares is exactly half of the total number of squares. Moreover, it turns out that our two possible pieces (the rectangle and the skew rectangle) always cover six white and six black squares, with one exception: 3×4-rectangles whose longer side is horizontal. These rectangles cover 4 white and 8 black squares, or vice versa. There has to be the same number of both kinds, since the number of white squares is the same as the number of black squares. Hence the number of rectangles in our covering whose longer side is horizontal has to be even, and the same holds (analogously) for rectangles whose longer side is vertical.

The same argument, applied to the following four (essentially identical) colourings (Figure 6.4) shows that the number of "skew" 3×4-rectangles has to be even as well. This, however, implies that the area of the large rectangle has to be divisible by 24, contradicting our assumption.

Summing up, we can say that a rectangle can be covered by hooks if it is of the form $k \times 12l$ ($k \neq 1, 2, 5$) or $3k \times 4l$. □

There are plenty of problems where colourings like these play a role. As a last example, let us consider the following:

Problem 57 25 pawns are placed in the lower left quarter of a 10×10-chessboard. Every pawn is allowed to jump over a pawn on an adjacent square (horizontally, vertically or diagonally) to an empty square. Is it possible to move all pawns to the upper left quarter (upper right quarter, respectively) by such jumps?

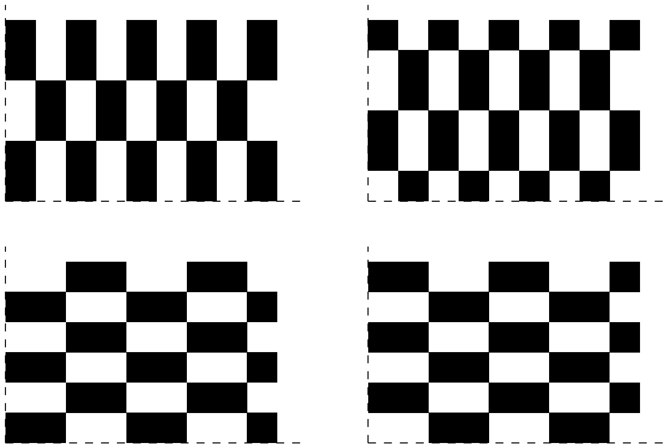

Figure 6.4: More ways to colour a rectangle.

Solution: First of all, let us consider the usual colouring of the chessboard, in which squares are alternatingly black and white. Without loss of generality, assume that the lower left corner is black. We notice that a pawn cannot change the colour of its square by any of the allowed jumps. In the beginning, there are 13 black squares and 12 white squares covered by pawns, whereas the upper left quarter consists of 12 black squares and 13 white squares. Hence it is impossible to move all the pawns there.

This argument, however, does not solve the problem for the upper right quarter. Does this mean that there is actually a possibility to move the pawns there? It turns out that the answer is still no, as can be seen by another colouring: black and white stripes (e.g., first row black, second row white, etc.). Once again, a pawn cannot change the colour of its square. In the beginning, there are 15 pawns on black squares, while it should be 10 at the end. Hence we obtain another contradiction, showing that the second task is also impossible. □

One of the most frequently used invariant is the parity (even or odd) of a parameter, as in the following two examples:

Problem 58 An urn contains exactly 2010 white and 2011 black balls. In addition to that, we have an unlimited supply of black balls. Now we repeatedly draw two balls randomly from the urn. If their colour is the same, then we remove them and add a black ball to the urn. If their colours differ, we return the white ball and remove the black ball. This is repeated until there is only one ball left. Determine the probability that the last ball is white.

Solution: In spite of the formulation of the problem, we do not actually have to compute any probabilities. Indeed, we will prove that the last ball has to be black. Notice first that the process always comes to an end, since the number of balls in the urn always decreases by 1. Let us now consider the number of white balls and prove that it is always even. In the beginning, it is an even number, namely 2010. If the two balls that we draw are both white, the number of white balls decreases by 2, otherwise it remains the same. Hence it stays even throughout the process.

At the end, the number of white balls has to be even. But if there is only one ball left, this means that this number has to be 0, which proves our assertion. □

Problem 59 Is it possible to arrange the numbers from 1 to 21 in a triangular table (the first row consists of six numbers, the second row of five numbers, etc.), such that every number from the second row on is the absolute difference of the two numbers above it? For comparison, the following table shows that this is possible with six numbers:

$$\begin{array}{ccccc} 6 & & 1 & & 4 \\ & 5 & & 3 & \\ & & 2 & & \end{array}$$

Solution: We consider the entries of the table modulo 2. Let the entries of the first row be $a_1, a_2, \ldots, a_6$. Modulo 2, addition and subtraction lead to the same result, and so we obtain the following table modulo 2:

- 1st row: a_1, a_2, a_3, a_4, a_5, a_6.
- 2nd row: $a_1 + a_2$, $a_2 + a_3$, $a_3 + a_4$, $a_4 + a_5$, $a_5 + a_6$.
- 3rd row: $a_1 + a_3$, $a_2 + a_4$, $a_3 + a_5$, $a_4 + a_6$.
- 4th row: $a_1 + a_2 + a_3 + a_4$, $a_2 + a_3 + a_4 + a_5$, $a_3 + a_4 + a_5 + a_6$.
- 5th row: $a_1 + a_5$, $a_2 + a_6$.
- 6th row: $a_1 + a_2 + a_5 + a_6$.

We add all entries and obtain a sum of

$$6a_1 + 8a_2 + 8a_3 + 8a_4 + 8a_5 + 6a_6 \equiv 0 \mod 2.$$

However, the sum of the 21 numbers is

$$\sum_{i=1}^{21} i = \frac{21 \cdot 22}{2} = 231 \equiv 1 \mod 2,$$

which shows that it is impossible to arrange them in such a table. □

Let us finally consider a problem that shows that often a fair amount of creativity is necessary to find appropriate invariants. Once the right invariant has been found, however, the solution is often remarkably short:

Problem 60 We start with the three numbers $0, 1, \sqrt{2}$, to which the following operation is repeatedly applied: one of the numbers is chosen and an arbitrary rational multiple of the difference of the two others is added. Is it possible to obtain the triple $0, \sqrt{2}-1, \sqrt{2}+1$ after a number of applications of this operation?

Solution: Since $\sqrt{2}$ is irrational, all numbers that are obtained during this process must have the form $a+b\sqrt{2}$ for some rational a and b. Such a number can be represented by the point (a, b) in the plane. We consider the triangle that is formed by the three numbers. In the beginning, its vertices are $(0, 0)$, $(1, 0)$ and $(0, 1)$. The described operation amounts to a translation of one of the points along a line parallel to the opposite side of the triangle. This operation does not change the area of the triangle, so the area remains constant. In the beginning, the area is $\frac{1}{2}$. The triangle that is formed by the three points $(0, 0)$, $(-1, 1)$ and $(1, 1)$, however, has area 1, so it is impossible to reach the triple $0, \sqrt{2}-1, \sqrt{2}+1$. $\square$

6.2 Potential functions

Let us now turn to potential functions. Such functions can often be used to prove that certain processes have to terminate. We have already seen a simple example in Problem 58 without paying too much attention to it. However, it is often necessary to introduce rather complicated parameters, as in the following two problems:

Problem 61 Each of the squares of an $n \times m$-table contains 1 or -1. We are allowed to multiply any row or column by -1 (i.e., reverse the signs of all entries). Prove that it is possible to repeatedly use this operation to reach a table whose row- and column sums are nonnegative.

Solution: We proceed in the following way: if the current table has a row or column whose sum is negative, we multiply its entries by -1. Of course, this can turn some other row- or column sums negative, so that it is not a priori clear that the process will eventually terminate. To prove that this is indeed the case, consider the sum of all entries of the table. This sum increases by at least 2 with each such operation, but it is obviously bounded above by mn (since each entry is at most 1). Hence the process has to terminate, which can only occur if there is no row or column whose sum is negative. $\square$

Problem 62 Twelve hermits are living in a forest, in houses that are either green or brown; some of them keep friendships. Each month of the year is the "special month" of one of the hermits. During the course of his special month, a hermit visits all his friends. If the houses of a majority (more than half) of his friends have a colour that differs from the colour of his own house, he changes the colour of his house. Prove: after a while, none of the hermits will have to change the colour of his house any more.

Solution: The approach is similar to the previous example. We consider the number of agreements, i.e., the number of pairs of friends whose houses have the same colour. Every time a hermit changes the colour of his house, the number of agreements increases according to the rules, since he now agrees with more than half of his friends. Since the number of agreements is bounded above (by the total number of pairs, which is $\binom{12}{2} = 66$), is cannot grow indefinitely, which means that the process has to come to an end, which happens when none of the hermits has a reason to change the colour of his house any more. □

6.3 Additional problems

Problem 63 Nine chess pieces are placed in the lower left corner of an 8×8-chessboard in such a way that they form a 3×3-square. A piece A is allowed to jump over any other piece B to the square that is the mirror image of its current square with respect to B's square. Is it possible to move the entire 3×3-square to the

1. upper left
2. upper right

corner of the chessboard?

Problem 64 Ten coins are placed in a circle in such a way that they all show "heads". Now we are allowed to perform the following operations:

- turn over four consecutive coins;
- turn over five consecutive coins, except for the middle one.

Is it possible to reach the configuration where all coins show "tails"?

Problem 65 A king moves on the chessboard in such a way that he visits all squares exactly once and returns to the original square. Prove that the number of diagonal moves on this tour has to be even.

Problem 66 From a triple $(a, b, c) \in \mathbb{R}^3$ we form a new triple according to the rule

$$(a, b, c) \to (b\sqrt{2} + c/\sqrt{2}, c\sqrt{2} + a/\sqrt{2}, a\sqrt{2} + b/\sqrt{2}).$$

Is it possible that we obtain the triple $(108 + 117\sqrt{2}, -196 - 54\sqrt{2}, 88 - 66\sqrt{2})$ from the triple $(4 - 21\sqrt{2}, 28 + 9\sqrt{2}, -32 + 12\sqrt{2})$ after a number of such steps?

Problem 67 The upper right quarter of the plane is divided into unit squares. Six squares in the corner are marked as shown in Figure 6.5. On each of them we place a coin. Now we are allowed to transform the position according to the following rule: if the squares above and to the right of a coin are empty, we may replace it by two coins on these squares. Is it possible to move all coins out of the marked area? Is it possible if there is only a coin on the corner square in the beginning?

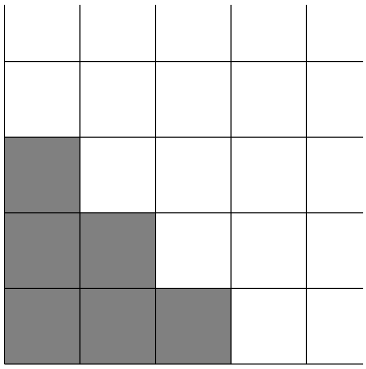

Figure 6.5: Six squares in the first quadrant of the plane

Problem 68 Let $k \geq 1$ be an integer. There is a fly on each square of a $(2k+1) \times (2k+1)$-board. Upon a signal, each fly moves to a diagonally adjacent square (it is allowed that more than one fly ends up on a square). Determine the minimum number of squares that remain empty now.

Problem 69 Prove: a $n \times m$-rectangle $(n, m > 1)$ can be filled completely with L-shaped pieces of the following form (Figure 6.6), where pieces are not allowed to overlap or to cover any area outside of the rectangle, if and only if nm is divisible by 8.

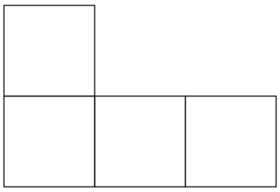

Figure 6.6: An L-tetromino

Problem 70 Is it possible to cover a 10×10-rectangle with 25 1×4-rectangles without gaps or overlaps?

Problem 71 (Canada 1994) 25 delegates at a meeting sit in a circle and vote in favour or against a certain proposal every hour. Delegates change their mind from one poll to the next whenever both neighbours' vote differed from theirs, otherwise their vote remains the same. Prove: after a while, no delegate changes their mind any more.

Chapter 7

Some concepts in graph theory

7.1 Definitions

Various graph-theoretical definitions and notations are used, and unfortunately they are not always consistent. Since we are mainly dealing with general concepts rather than formal definitions here, we will only introduce some intuitive notions.

A *graph* consists of a set V of *vertices* and a set E of *edges*, where each of the edges connects two vertices v_1 and v_2. We write such an edge as a pair (v_1, v_2) of vertices (in this context, (v_1, v_2) and (v_2, v_2) are considered to be the same edge; if they are considered distinct, one says that the graph is *directed*). An edge whose two ends are identical is called a *loop*. It is also possible that a pair of vertices is connected by several edges (*multiedges*). A graph that contains loops and/or multiedges is called a *multigraph*.

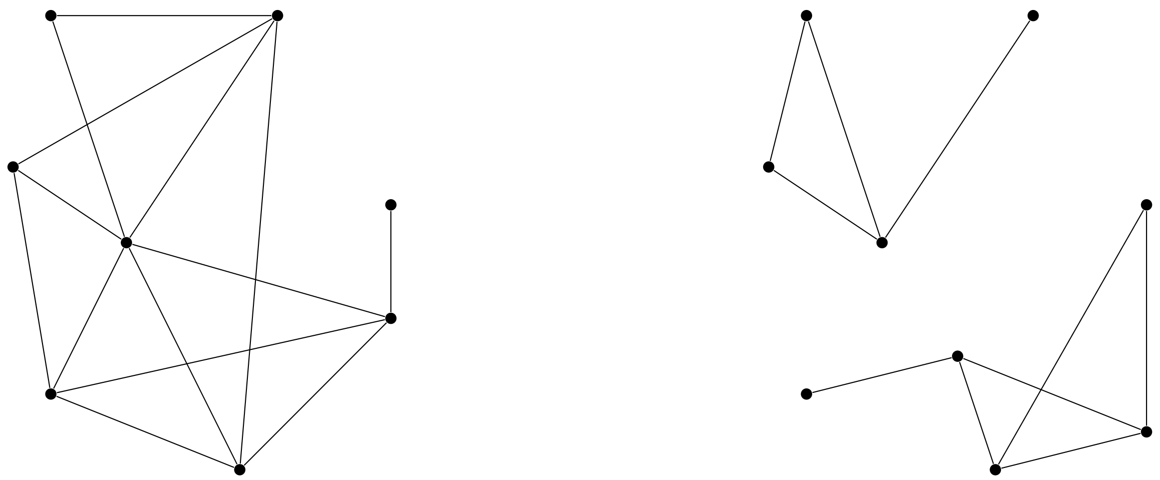

Figure 7.1: Two graphs: the left graph is connected, the right one is not.

Intuitively, one can think of points in the plane that are joined by line segments or generally curves, as in Figure 7.1. An important and also quite intuitive concept is connectedness of graphs: a graph is called *connected* if any two vertices are connected by a *path* (a continuous sequence of edges between the two vertices). For instance, the graph on the left of Figure 7.1 is connected, the right one is not.

The connected parts of a (disconnected) graph are called its (connected) *components*. The set of vertices that are connected to a given vertex v by an edge is called the *neighbourhood* of v. The number of edges going out from a vertex is the *degree* $\deg v$ of v. A loop is counted twice in the degree. The most famous result on the degrees of a graph is the so-called *handshake lemma*:

Theorem 7.1 (Handshake lemma) *The sum of the degrees in a (multi-)graph is twice the number of edges. In particular, the number of vertices of odd degree is even.*

Proof: The degree of a vertex v is equal to the number of edges of which v is an endpoint. Hence the sum of all degrees is also the total number of endpoints, and since each edge has two ends, the first part follows immediately. Hence the sum of all degrees is an even number, which is only possible if the number of vertices with odd degree is even. $\square$

The reason for the name "handshake lemma" lies in the following interpretation: at a party, some of the guests shake hands. If one counts the number of handshakes per person, then the sum of these values is precisely twice the total number of handshakes.

The following problem is essentially a disguised version of the handshake lemma. However, it shows that a graph-theoretical interpretation can often be very useful. One of the main points in solving a combinatorial problem is often to formulate the question in a suitable mathematical way. The representation by a graph is often advantageous in this context.

Problem 72 A number of straight lines are drawn on a piece of paper. Proof that the number of lines that intersect an odd number of lines at an angle $< 30°$ is even.

Solution: We assign a vertex of a graph to each of the lines, and we draw an edge between any two vertices if and only if the corresponding lines intersect at an angle $< 30°$. Then it becomes evident that the statement of the problem is exactly the second part of the handshake lemma. $\square$

7.2 Eulerian and Hamiltonian cycles

In this section, we discuss two important concepts of graph theory: Eulerian cycles and Hamiltonian cycles. The famous problem of the seven bridges of Königsberg (Figure 7.2) that was treated by Euler is considered to be one of the oldest problems in graph theory: in the city of Königsberg (now Kaliningrad in Russia), there were seven bridges over the Pregel river. The question was whether one could find a walk using each of the bridges exactly once.

We represent the bridges as edges in a multigraph, as in Figure 7.3. In graph-theoretical terms, the question can now be formulated as follows:

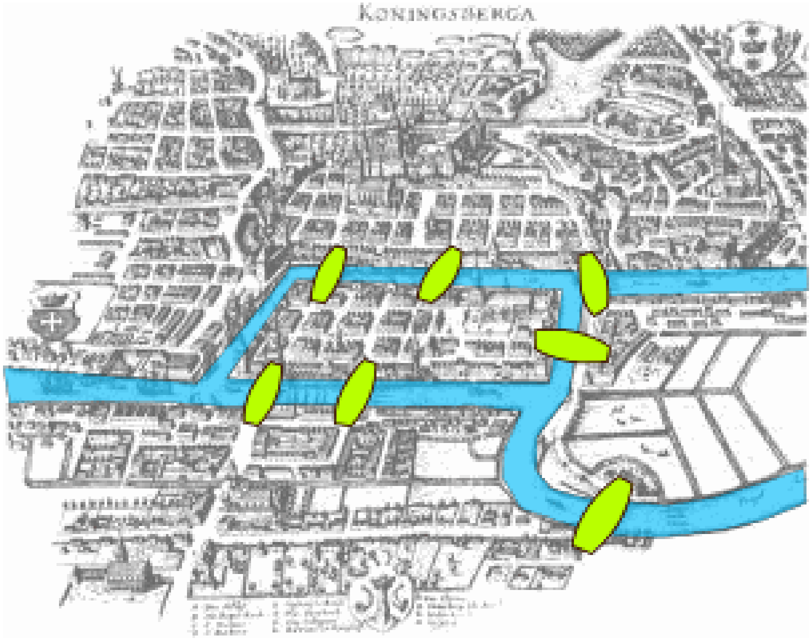

Figure 7.2: The seven bridges of Königsberg

Definition 7.2 An *Eulerian path* (or *trail*) is a path in which each edge of the graph is used exactly once. An *Eulerian cycle* is an Eulerian path in which the first and last edge also share an endpoint (i.e., the path is closed).

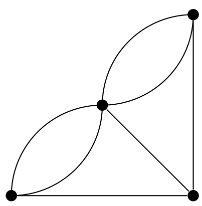

Figure 7.3: The seven bridges of Königsberg: graphical representation

Theorem 7.3 *A connected (multi-)graph has an Eulerian path if and only if at most two of the vertices have odd degree. It has an Eulerian cycle if and only if there are no vertices of odd degree.*

Proof: To prove that the condition is necessary, we make use of an invariant. It is clear that an Eulerian path that enters a certain vertex through some edge has to leave it through another edge. Exactly two edges ending in a given vertex are used in this process. Hence the number of edges in an Eulerian path of which a fixed vertex v is an endpoint (and this has to be the number of all edges of which v is an endpoint) has to be even. The only possible exceptions are the beginning and the end of the Eulerian path. In the case of Eulerian cycles, there can be no exceptions.

It remains to prove that the condition is also sufficient. Let us consider Eulerian cycles first, i.e., we assume that all vertex degrees are even. Consider the longest possible path that does not use any edge twice. Since it cannot be extended, all edges going out from the end vertex of our path are already used. By our assumption, this is an even number. By the above argument, this is only possible if the end vertex coincides with the initial vertex of our path, hence the path forms a cycle.

Assume that this cycle is not an Eulerian cycle. Then there have to be unused edges, and since the graph is connected, there is at least one among these edges that has one of the vertices on the cycle as an end. If we start with this edge and continue on the cycle, we obtain a longer path that uses each edge at most once, which is a contradiction. Hence the proof is complete in the case of Eulerian cycles.

If there are exactly two vertices of odd degree, then we add an edge between the two. The resulting graph then satisfies the condition for an Eulerian cycle. Removing this artificial edge from the Eulerian cycle, we obtain an Eulerian path in the original graph.

□

REMARK: In the problem of the seven bridges of Königsberg, there is no walk of the desired form, since all four vertices in the corresponding multigraph have odd degree.

Definition 7.4 A *Hamiltonian path* is a path in a graph that visits each vertex exactly once. A *Hamiltonian cycle* is a closed path that visits each vertex exactly once.

Theorem 7.5 (Dirac's Theorem) *If every vertex in a graph with $n \geq 3$ vertices has degree $\geq n/2$, then the graph has a Hamiltonian cycle.*

Proof: Note first that the graph has to be connected: otherwise, the smallest connected component would have $\leq n/2$ vertices, and each of the vertices in this component could have at most $n/2 - 1$ neighbours, which contradicts our assumption.

Let us now consider the longest path that does not visit a vertex twice, and let the vertices on this path be $v_1, v_2, \ldots, v_m$. Since the path is maximal, all neighbours of v_1 and v_m have to lie on the path, since one could otherwise extend it. There are $\deg v_1 \geq n/2$ values of i ($2 \leq i \leq m$) for which (v_1, v_i) is an edge, and there are $\deg v_2 \geq n/2$ values of i ($2 \leq i \leq m$) for which (v_{i-1}, v_m) is an edge. Their sum is at least $n \geq m > m-1$, which means that by the pigeonhole principle there is a value i for which both assertions hold. Thus there exists a closed path $v_1, v_2, \ldots, v_{i-1}, v_m, v_{m-1}, \ldots, v_i, v_1$ (Figure 7.4). However, this closed path has to be a Hamiltonian cycle: since the graph is connected, there would otherwise have to be a vertex w that is a neighbour of a vertex v_k of the cycle. If the edge (v_k, w) is added to the cycle, one obtains a longer path that does not visit any vertex more than once, contradicting our assumption. □

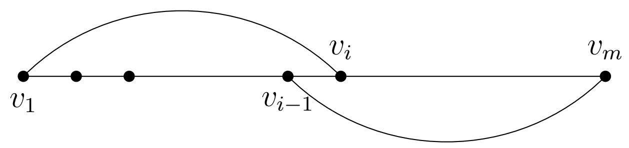

Figure 7.4: Proof of Theorem 7.5

7.3 Plane graphs and Euler's polyhedron formula

Definition 7.6 A *plane graph* is a graph whose vertices are points in the plane, and whose edges are curves connecting the two endpoints. Moreover, it is required that the edges do not intersect (except at their endpoints). Figure 7.5 shows a plane graph. The edges of the graph divide the plane into regions that are called *faces*.

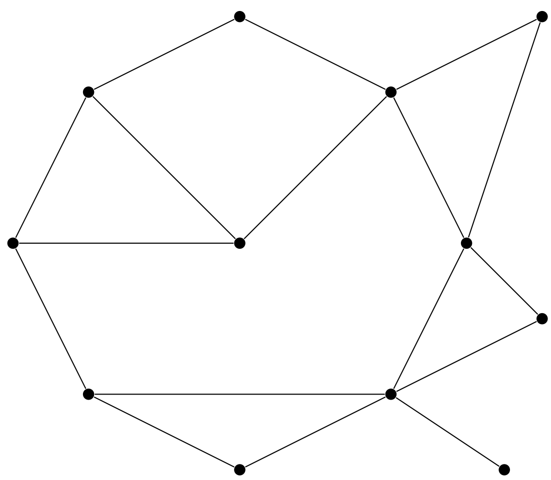

Figure 7.5: A plane graph.

Examples of plane graphs are the nets of polyhedra, such as tetrahedron or cube in Figure 7.6. In these cases, the faces are indeed the faces of the polyhedron. The following classical result on plane graphs is known as *Euler's polyhedron formula.*

Theorem 7.7 (Euler's polyhedron formula) *Let e be the number of vertices, k the number of edges and f the number of faces of a plane graph, and let c be the number of its connected components. Then one has*

$$e - k + f = c + 1.$$

In particular, for connected plane graphs (and thus for polyhedra), the formula $e-k+f = 2$ holds.

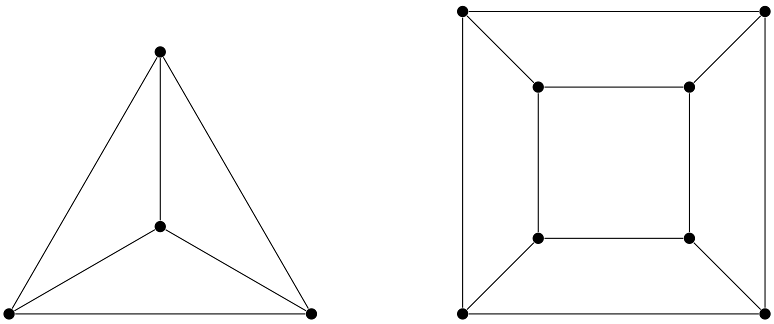

Figure 7.6: Nets of tetrahedron and cube.

Proof: We prove the formula by induction on the number of edges. If this number is equal to 0, then each vertex is also a connected component, and so we have $e = c$, $f = 1$ and $k = 0$, which means that the formula is trivially satisfied. For the induction step, we have to distinguish two cases:

- An additional edge connects two components (Figure 7.7, left). In this case, e and f remain the same, k increases by 1, while c decreases by 1. Thus the formula remains true.
- An additional edge connects two vertices that belong to the same component. In this case a face is split into two (Figure 7.7, right). Then k and f increase by 1, e and c stay the same. Again, the formula remains correct.

This completes the induction. □

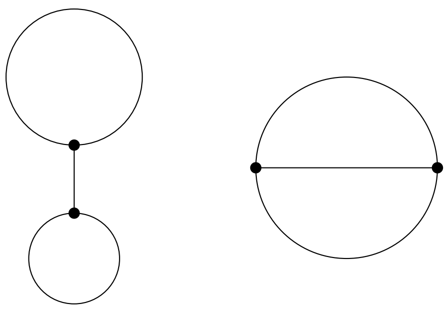

Figure 7.7: Proof of Theorem 7.7

Problem 73 Is it possible to connect five buildings pairwise by paths such that no two paths cross (bridges, tunnels, etc. are forbidden)?

Solution: Suppose this was possible. Then the buildings and paths have to form a plane graph with 5 vertices and $\binom{5}{2} = 10$ edges. Since this graph is connected, it must have $10 + 2 - 5 = 7$ faces by the polyhedron formula. Each of these faces has at least three edges on its boundary, and each edge belongs to at most two distinct faces. It follows that the number of edges is at least $\frac{3f}{2}$. In this case, however, $\frac{3f}{2} \geq \frac{21}{2} > 10$, a contradiction. This means that such a configuration is impossible.

7.4 Ramsey numbers

Here we continue Problem 43. More generally, we consider a *complete graph* (i.e., all vertices are pairwise connected by edges) whose edges are coloured with k different colours. We claim that for a sufficiently large number of colours there is always a *monochromatic triangle*, i.e., three vertices whose connecting edges are coloured with the same colour. In our first example, "knowing – not knowing" corresponds to two different colours, in the second example the three topics correspond to three different colours.

We define the sequence n_k by $n_1 = 3$ and $n_k = kn_{k-1} - k + 2$. Is can be shown that $n_k = \lfloor k!e \rfloor + 1$ ($e = 2.71828\ldots$) holds, but this is not essential for our purposes. We prove by means of induction that a complete graph with n_k vertices whose edges are coloured with k different colours always contains a monochromatic triangle. For $k = 1$, this is trivial. For the induction step we consider an arbitrary vertex. Since it has $kn_{k-1} - k + 1 = k(n_{k-1} - 1) + 1$ neighbours, the pigeonhole principle shows that it is connected to n_{k-1} of its neighbours by edges of a single colour. If there are two vertices among these neighbours that are also connected by an edge of this colour, we have already found a monochromatic triangle. Otherwise, only $k - 1$ colours remain for the edges between these n_{k-1} vertices, so that the statement follows from the induction hypothesis. □

More generally, the following theorem holds:

Theorem 7.8 (Ramsey's theorem) *For arbitrary natural numbers k, r, there is a natural number n such that a complete graph with n vertices whose edges are coloured with k different colours always contains a set of r vertices with the property that the edges between these vertices are all coloured with the same colour.*

7.5 Additional problems

Problem 74 The participants of an olympiad camp play a lot of table tennis. Prove that there are two participants who played against the same number of different opponents.

Problem 75 Any two of the knights at King Arthur's round table are either friends or enemies. However, each knight has more friends than enemies. Prove that Arthur can arrange his men around the table in such a way that none of them sits next to one of his enemies.

Problem 76 Nine points are given in the plane, no three of them lie on a line. We draw all the line segments between these points, some of them in blue, the others in green. Each of the triangles formed by three of the points has at least one blue side. Prove that there are four points such that all the segments between these points are blue.

Problem 77 During a fierce discussion among the 400 members of the South African National Assembly, each of the members insults exactly one of the others. Later, a committee of 134 members is to be formed to investigate the incident. Prove that it is possible to select this committee in such a way that none of the committee members was insulted by another committee member.

Problem 78 Three buildings should be connected to the water reservoir, the power station and the sewage plant. Is it possible to do this in such a way that the pipes/cables do not cross anywhere?

Problem 79 Ten airline companies offer flights between 2011 different cities. Between any two cities, there are flights offered in both directions by at least one of the companies. Prove that one can book a round trip with one of the companies that visits an odd number of cities.

Chapter 8

Various

8.1 Induction proofs in combinatorics

We have already used induction in a number of proofs. Induction is much more than a method to prove certain identities or inequalities for all natural numbers. It is thus not surprising that many olympiad problems with a combinatorial flavour can be solved by means of induction. In the following example, for instance, an inductive approach simplifies the argument:

Problem 80 On an arbitrarily large rectangular table, n cells are given. Prove that it is possible to colour these cells with two colours (red and blue) so that in each row or column the number of red squares differs from the number of blue squares by at most 1.

Solution: The statement is trivial for $n = 0$ or $n = 1$ (the word “trivial” should be used with great care, but here it might be appropriate). For larger values of n, we distinguish three cases:

1. Four of the given cells form a rectangle. Then we colour all the other cells according to our rule (which is possible by the induction hypothesis) and colour two diagonally opposite cells of the rectangle red and the others blue, which yields a feasible colouring.

2. Three of the given cells are the corners of a rectangle (the fourth is not among the given cells). Remove these three cells, add the fourth corner of the rectangle instead, and colour the cells according to the induction hypothesis. If the added cell is red (blue), we colour the adjacent corners of the rectangle red (blue) and the opposite corner blue (red), which again yields a feasible colouring.

3. If none of the above cases applies, choose any row or column that contains more than one given cell (if there is no such row or column, we can colour the cells arbitrarily). The squares of this row (column) are coloured in such a way that the number of red cells differs from the number of blue cells by at most 1. All the other cells are

unaffected, so that we can colour them in a feasible way according to the induction hypothesis again.

□

Quite often one can use induction to prove general conjectures, as in the following problem:

Problem 81 We are given ten different weights whose masses are $1, 2, 4, \ldots, 512$ grams. An object of integer mass M is weighed by putting it together with the ten weights on the two pans of a balance scale such that we reach an equilibrium.

- Prove that no object can be weighted in more than 89 ways.
- Give an example for M such that there are precisely 89 ways to weigh the object.

Solution: Instead of solving the original question, let us first try to consider cases with less weights: if there is only one weight of mass 1, it is only possible to weigh objects with a mass of 1, and there is only one way. If one has two weights with masses 1 and 2, then it is already possible to weigh an object of mass 1 in two ways, and no object in more than two ways. If we have three weights with masses 1,2 and 4, an object of mass 1 or 3 can be weighed in three ways. One could now come to the conjecture (possibly by considering some more cases as well) that the maximum number of ways to weigh an object (denoted by w_n) with the n weights $1, 2, \ldots, 2^{n-1}$ is exactly the Fibonacci number f_{n+1} ($f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$). For $n = 1$ and $n = 2$, this is obviously true.

Let us now proceed with the induction step, where we have to consider two cases regarding the object that is to be weighted:

- The object has an even mass. In this case, we cannot use the weight with mass 1 (since all the other weights have even mass), and we are left with $n-1$ weights whose masses are $2, 4, \ldots, 2^{n-1}$. Dividing all masses by 2 (which amounts to choosing a new unit), we see that the problem is reduced to the case of $n-1$ weights, which means that there are at most w_{n-1} possibilities.

- The object has odd mass M. Then there are two possibilities for the 1-gram weight: it can be on the same scale of the balance as the object or on the other scale. Then we are left with the problem of weighing $M \pm 1$ with weights of masses $2, 4, \ldots, 2^{n-1}$. One of the two numbers $M+1, M-1$ is divisible by 4, the other one is not. In one of the cases, we divide all masses by 4 (the 2-gram weight cannot be used), in the other case by 2. Once again, we have reduced the problem to simpler cases, namely to those with $n-1$ and $n-2$ weights respectively. Hence the maximum number of ways to weigh an object is at most $w_{n-1} + w_{n-2}$, which shows that the Fibonacci numbers are indeed an upper bound.

Finally, we have to determine a mass M that can indeed be weighed in f_{n+1} ways. We prove by induction that

$$M = \begin{cases} \frac{2^n+1}{3} & n \text{ odd} \\ \frac{2^n-1}{3} & n \text{ even} \end{cases}$$

satisfies this condition. For $n = 1$ and $n = 2$ this is again easy to see. The induction step uses the above argument again. In the first case (n odd), we have $M = \frac{2^n+1}{3}$, and $M+1 = 4 \cdot \frac{2^{n-2}+1}{3}$ is divisible by 4. Hence if the 1-gram weight is placed on the same scale as the object, we have f_{n-1} possibilities according to the induction hypothesis.

On the other hand, $M - 1 = 2 \cdot \frac{2^{n-1}-1}{3}$, so that we have f_n possible ways to weigh the object if the 1-gram weight is placed on the other scale. Altogether, we obtain $f_n + f_{n-1} = f_{n+1}$ possibilities. The second case (n even) can be completed analogously.

In our concrete case, the Fibonacci number f_{11} is equal to 89, and an object of mass $\frac{2^{10}-1}{3} = 341$ grams can be weighed in 89 ways. □

8.2 Combinatorial geometry

There are many interesting combinatorial problems dealing with geometric objects. We have already seen a couple of examples. Since it is hard to give general guidelines for the solution of problems in combinatorial geometry, we only consider two examples here.

Problem 82 We know that n given points in the plane have the property that none of the triangles that is formed by three of the points has an area that is greater than 1. Prove that there is a triangle whose area is at most 4 that contains all n points.

Solution: This problem is an excellent example of the so-called *extremal principle* that is generally very useful. The basic idea is to select the largest or smallest object among a given set: in this problem, we consider those three points among the given set for which the triangle that is formed by the points has maximal area. Let this triangle be ABC. By the given assumption, the area of ABC is at most 1. None of the other points can be further away from AB than C, since this would imply the existence of a larger triangle. Likewise, none of the points is further away from AC than B or further away from BC than A. This means that all points lie inside the triangle that is formed by the three lines that pass through one of the corners of ABC and are parallel to the line through the two other corners. This triangle is similar to ABC and can be obtained from ABC by a homothety with a ratio of -2, so its area is ≤ 4 (Figure 8.1). □

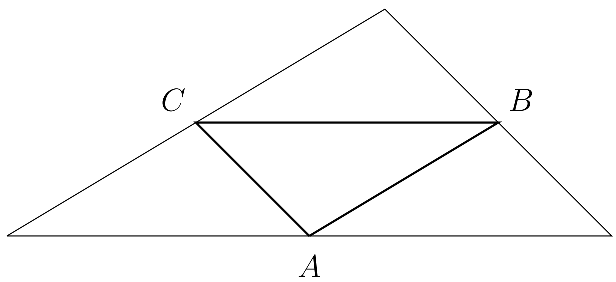

Figure 8.1: The triangle ABC and the region that contains all the points.

Problem 83 (Iran 1999) Let ABC be an arbitrary triangle. Prove: if all points of the plane are coloured either red or green, then there are either two red points whose distance is 1 or three green points that form a triangle that is congruent to ABC.

Solution: Let $a \leq b \leq c$ be the side lengths of a triangle ABC. We assume that there are neither two red points with distance 1 nor three green points that form a triangle that is congruent to ABC.

Let us first suppose that there are two red points M and N whose distance is a. Furthermore, let P be chosen in such a way that PMN is congruent to ABC. The circles around M and N of radius 1 are completely green, since there would otherwise be two red points whose distance is 1. Hence the circle around P of radius 1 is completely red, since we would otherwise obtain a green triangle that is congruent to ABC and results from a simple translation of PMN. However, there are two points on this circle whose distance is 1 (forming an equilateral triangle with P), contradiction.

Hence there are no two red points whose distance is a. If there are no red points at all, the statement is trivial. Hence we can choose a red point R and consider the circle of radius a around R. This circle has to be completely green. For each point S on this circle, there is another point T on the circle that forms an equilateral triangle with R and S. Since a is the shortest side of the given triangle, there is also a point U outside of the circle around R such that UST is congruent to ABC. Hence this point U has to be red. If we let S rotate on the circle around R, then U traces a circle whose radius is greater than a. All points on this circle have to be red. However, there are two points on this circle whose distance is a, and we obtain a contradiction. $\square$

8.3 Mathematical games

Sometimes one encounters problems in olympiad competitions in which two (sometimes more than two) players play a certain game and strategies for the two players are sought. The typical question is: which player wins if both play optimally? Two ideas are particularly important: noticing symmetries (which is generally an important principle in combinatorics) and finding winning positions. In order to explain these ideas, let us consider one problem of each type.

Problem 84 Two players play an $(O - X)$-game on a 10×10 board according to the following rules: taking turns, they mark the squares with their own symbol (X begins). At the end, they determine the number A, which counts the number of times that five X's occur in a row (horizontal, vertical or diagonal), and the number B, which counts the number of times that five O's occur in a row. Six X's in a row count as two points, seven X's count as three points, etc. The first player wins if $A > B$, the second player if $A < B$. The game is declared a draw if $A = B$. Is there a strategy for the first player that guarantees him (a) a win (b) at least a draw?

Solution: This problem can be solved by means of a simple symmetry argument: the second player can always force a draw by taking the mirror image (with respect to the centre) of every move by his opponent. In this way, the position at the end has to be symmetric, which implies that $A = B$, since every row of five X's corresponds to a row of five O's.

On the other hand, the first player can easily avoid a loss. This is generally true for all games with similar symmetric rules, provided that it cannot be a disadvantage to make a move (which clearly is the case here): it suffices to place the first X randomly, then to take the mirror image of all the opponent's moves, if this is possible. If not, one can again play randomly. In this way, the first player can also enforce a symmetric final position, which means that any game has to end in a draw if both players use an optimal strategy. $\square$

Searching for winning positions is particularly helpful if the players take turns in their moves and have the same possibilities in any given position. We define a *winning position* as a position in a game that is lost by the player whose turn it is to move (if both players play in the best possible way). In other words, it is advantageous to leave a winning position after one's move. All other positions are called *losing positions*. The following rules (that are very easy to justify) hold:

- A position is a losing position if and only if there is a move that leads to a winning position (such a move should obviously be played).
- A position is a winning position if and only if every possible move leads to a losing position (otherwise, the player to move would have a way to win the game).

This defines winning positions in a recursive way. The rules of the game determine whether the possible final positions are winning or losing positions.

Problem 85 (Italy 1999) Alice and Bob play the following game: there are 1999 balls on a table. The players take turns to remove some of the balls (Bob first), the only restriction being that one has to take at least one ball and at most half of the balls. A player who leaves only one ball behind, loses the game. Which player has a winning strategy?

Solution: In this simple game, a position is given by the number of remaining balls. We have to determine for which numbers n the position with n remaining balls is a winning position. According to the rules, $n = 1$ is a losing position. This implies that $n = 2$ is a winning position, since there is only one possible move. For $n = 3$ and $n = 4$, one can make a move that leaves only two balls behind, hence they are losing positions. On the other hand, $n = 5$ is a winning position again. Generally, we show that the winning positions are precisely those of the form $n = 3 \cdot 2^k - 1$ (all others being losing positions). Since 1999 is not of this form, Bob wins (by removing 464 balls, which yields the winning position $3 \cdot 2^9 - 1 = 1535$).

We prove this formula by induction. For $k = 0$, there is nothing left to prove. We assume that $3 \cdot 2^k - 1$ is a winning position. Now one can remove an appropriate number of balls from each of the positions $n = 3 \cdot 2^k, 3 \cdot 2^k + 1, \ldots, 3 \cdot 2^{k+1} - 2$, so that only $3 \cdot 2^k - 1$

remain, thus reaching a winning position. Hence all of these positions are losing positions. On the other hand, such a move is no longer possible for $n = 3 \cdot 2^{k+1} - 1$, since one would have to remove more than half of the balls. Thus one has to move to a losing position. This means that $n = 3 \cdot 2^{k+1} - 1$ is indeed a winning position, which completes the induction.

□

8.4 The principle of double counting

A very elegant principle that can be applied to many combinatorial problems is the idea of *double counting*. Here, a certain quantity is counted in two different ways and the results compared so as to obtain an identity. A variant of this principle is to determine upper and lower bounds for the same quantity and to compare them. A simple example to illustrate the main idea is a table whose entries are numbers. Then the sum of all the entries can be determined in two ways: by adding up the row sums, or by adding up the column sums. Obviously, the result of these two operations should be the same.

Let us now discuss two problems to which this idea can be applied:

Problem 86 (IMO Shortlist 2004) There are 10001 students at a university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:

- Each pair of students are in exactly one club.
- For each student and each society, the student is in exactly one club of the society.
- Each club has an odd number of students. In addition, a club with $2m + 1$ students (m is a positive integer) is in exactly m societies.

Find all possible values of k.

Solution: Let us consider the situation as an organigram, in which the societies are placed on top, the clubs in the middle and the students at the bottom. Lines are drawn between societies and the clubs that belong to these societies, and between clubs and their members. We determine the number of connections between students and societies in two different ways:

- Since each student is a member of exactly one club of each society, there is exactly one connection for each student-society pair. Hence the number is exactly $10001k$.
- Each club with $2m + 1$ members creates precisely $m(2m + 1)$ connections. Hence if $2m_1 + 1, 2m_2 + 1, \ldots, 2m_r + 1$ denote the number of members in each club, the total number of connections is
$$\sum_{i=1}^{r} m_i(2m_i + 1).$$

These two values have to be the same. On the other hand, let us count the number of pairs of students in two ways: on the one hand, there are obviously $\binom{10001}{2}$ pairs. On the other hand, each pair belongs to exactly one common club, and a club with $2m+1$ members gives rise to exactly $\binom{2m+1}{2}$ pairs. Hence we have

$$\sum_{i=1}^{r} \binom{2m_i+1}{2} = \binom{10001}{2}.$$

Taking into account that $\binom{2m+1}{2} = m(2m+1)$, we find

$$10001k = \sum_{i=1}^{r} m_i(2m_i+1) = \sum_{i=1}^{r} \binom{2m_i+1}{2} = \binom{10001}{2}$$

or $k = \frac{1}{10001}\binom{10001}{2} = 5000$. □

Problem 87 (Taiwan 2000) Let S be the set $\{1, \ldots, 100\}$, and let P denote the family of all 49-element subsets of S. To each set T in P, we associate a number $n(T)$ in S. Prove that one can select a 50-element subset M of S in such a way that $n(M \setminus \{x\}) \neq x$ for all $x \in M$.

Solution: The number of 50-element subsets is $\binom{100}{50}$. If we can show that the number a of "forbidden" subsets (i.e., those for which there is an $x \in M$ with $n(M \setminus \{x\}) = x$) is less than $\binom{100}{50}$, we are done. Now note that for every set $T \in P$, there is at most one forbidden set, namely $T \cup \{n(T)\}$ (if $n(T) \notin T$, so that this set really contains 50 elements). The number of forbidden sets is thus at most equal to the number of elements in P, which is $\binom{100}{49}$.However, we have

$$\binom{100}{49} = \frac{50}{51}\binom{100}{50} < \binom{100}{50},$$

which proves the statement. Note here that generally, the numbers in Pascal's triangle increase until the middle binomial coefficient, and then decrease again. □

8.5 Additional problems

Problem 88 A game board consists of n squares in a row. There is a token on each of the squares. In the first move one can move any token to one of the neighbouring squares, creating a pile of two tokens (hence there are two possible moves with each token at the beginning, except for those at the two ends). Thereafter, one may move any pile of tokens by as many squares as there are tokens in the pile (this includes piles consisting of a single token). If a pile lands on another pile, they are merged. Prove that it is possible to merge all tokens to a single pile in $n-1$ moves.

Problem 89 We choose $2^n - 2$ $(n > 1)$ subsets of a finite set M of numbers, each of which contains at least half of the elements of M. Prove that one can select $n-1$ elements of M in such a way that each of the given subsets contains at least one of these elements.

Problem 90 (IMO 1997) An $n \times n$-matrix whose entries come from the set $S = \{1, 2, \dots, 2n-1\}$ is called *silver matrix* if, for each $i = 1, 2, \dots, n$, the ith row and the ith column together contain all elements of S. Show that:

1. there is no silver matrix for $n = 1997$;
2. silver matrices exist for infinitely many values of n.

Problem 91 Prove: there exists a configuration of $n(n+1)$ points in the plane such that for each k with $1 \le k \le 2n$, there is a straight line that passes through exactly k of the points.

Problem 92

a. Determine the smallest number of lines that one can draw on a 3×3-chessboard in such a way that each square of the board has an inner point that lies on one of the lines. Draw a minimal configuration and prove that the answer is indeed a minimum.

b. Analogous to a. for a 4×4-board.

Problem 93 (Germany 2005) We are given n circles ($n \ge 2$) in the plane, such that each circle cuts each other circle twice and all these points of intersection are pairwise distinct. Each point of intersection is coloured with one of n colours in such a way that each colour is used at least once and the same number of colours k is present on each of the circles. Determine all values of n and k for which such a colouring is possible.

Problem 94 (IMO Shortlist 2000) Ten gangsters are standing on a flat surface, and the distances between them are all distinct. At twelve o'clock, when the church bells start chiming, each of them shoots at the one among the other nine gangsters who is the nearest. How many gangsters will be killed at least?

Problem 95 The following game is played on an ordinary 8×8-chessboard: the first player places a queen on a square on the right or upper boundary of the board. Thereafter, the players take turns to move the queen, either to the left, or down, or diagonally to the left and down, by an arbitrary number of squares. The winner is the one who reaches the lower left corner. Which player wins this game if both apply a perfect strategy?

Problem 96 A chocolate bar consists of $9 \times 6 = 54$ pieces. Two players share this bar according to the following rules: taking turns, they are allowed to break off one row and eat it (at the beginning, for instance, the first player can either take nine pieces and leave a 9×5-bar, or take six pieces and leave a 8×6-bar). If at any time, a $2 \times m$-bar is divided into two $1 \times m$-bars, both players get one of these pieces. Prove that the first player has a strategy that guarantees him at least six pieces of chocolate more than his opponent. Can he improve on this strategy, i.e., always get more than 30 pieces?

Problem 97 (IMO Shortlist 1994) On a 5×5-board, two players alternately mark numbers on empty cells. The first player always marks 1's, the second 0's. One number is marked per turn, until the board is filled. For each of the nine 3×3-squares, the sum of the nine numbers on its cells is computed. Denote by A the maximum of these sums. How large can the first player make A, regardless of the responses of the second player?

Problem 98 (IMO Shortlist 2004) Let N be a positive integer. Two players A and B, taking turns, write numbers from the set $\{1, \ldots, N\}$ on a blackboard. A begins the game by writing 1 on his first move. Then, if a player has written n on a certain move, his adversary is allowed to write $n+1$ or $2n$ (provided the number he writes does not exceed N). The player who writes N wins. We say that N is of type A or of type B according as A or B has a winning strategy.

1. Determine whether $N = 2004$ is of type A or of type B.
2. Find the least $N > 2004$ whose type is different from the one of 2004.

Problem 99 (Baltic Way 2004) A and B play the following game: starting from a pile of $n \geq 4$ stones, the players take turns to select a remaining pile and divide it into two smaller (nonempty) piles. A begins. The first player who cannot make a move any more, loses. For which values of n does A have a winning strategy?

Problem 100 (Italy 1999) Let X be a set of n elements and $A_1, \ldots, A_m$ be subsets of X with

- $|A_i| = 3$, $i = 1, \ldots, m$.
- $|A_i \cap A_j| \leq 1$ for every pair $i \neq j$.

Prove that there exists a subset of X with at least $\lfloor \sqrt{2n} \rfloor$ elements that does not contain any A_i as a subset.

Problem 101 (IMO 1998) In a competition there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that for any two judges their ratings coincide for at most k contestants. Prove: $k/a \geq (b-1)/(2b)$.

Problem 102 (IMO 2001) Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that each contestant solved at most six problems, and for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy. Show that there is a problem that was solved by at least three girls and at least three boys.

Bibliography

[1] M. Aigner. *Discrete Mathematics.* American Mathematical Society, Providence, RI, 2007.

[2] R. Bodendiek and G. Burosch. *Streifzüge durch die Kombinatorik.* Spektrum Akademischer Verlag, Heidelberg, 1995.

[3] D. Fomin, S. Genkin, and I. Itenberg. *Mathematical circles (Russian experience)*, volume 7 of *Mathematical World.* American Mathematical Society, Providence, RI, 1996.

[4] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete mathematics.* Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, 1989.

[5] M. S. Klamkin. *International Mathematical Olympiads 1978–1985 and forty supplementary problems*, volume 31 of *New Mathematical Library.* Mathematical Association of America, Washington, DC, third edition, 1999.