A TASTE OF GEOMETRY

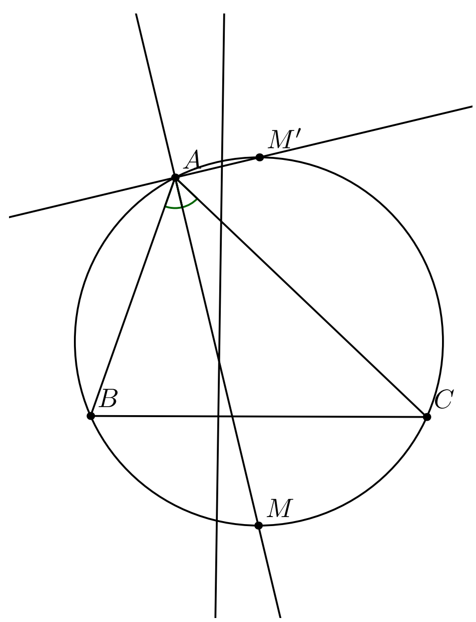
1. Welcome!

Hi friends,welcome to MATHOMETER a site we intend as a collection of resources created by olympiad enthusiasts just like you. We will try our best to keep you updated with fascinating ideas that you will find useful in your encounter with olympiad problems. Hope you enjoy your every visit here!!

2. a nice lemma

So Today we will learn about the following lemma. Seeing the lemma,you might think that its too simple,but sometimes they are the simple ideas which act as a bridge leading to non trivial results(No other example can be as famous as the Dirichlets pigeonhole principle). We hope you will get convinced by the applications of the lemma that will follow. So let us Begin!

Lemma 1. *Let ABC be a triangle with $AB \neq BC$. Draw the interior(exterior)angle bisector of A and the perpendicular bisector of BC.Let them meet at $M(M')$. Then $M(M')$ lies on the circumcircle of ABC*



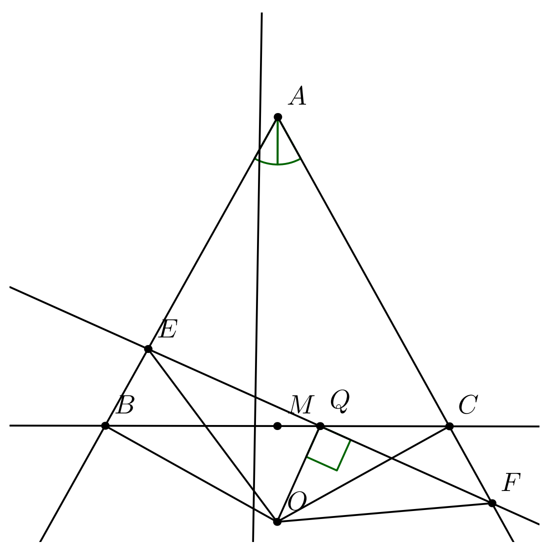
Date: 23 January 2016.

Proof. Of course the proof is not at all difficult.Just consider a point $M(M')$ which is midpoint of minor(major) arc BC.Now unless $AB = AC$, this point $M(M')$ will lie on the perpendicular bisector of BC as well as angle bisector of A,which was to be proved. □

So yes,we are all set to use our lemma in the problems,only thing is that do not forget to check the case separately where the triangle is isosceles!.

3. Problems

Problem 1. *Problem 1(IMO 1994 P2) ABC is an isosceles triangle with $AB = AC$.m is the midpoint of BC and O is the point on line AM such that $OB \perp AB$.Q is an arbitrary point on BC different from B and C.E lies on AB and F lies on AC such that E, Q, F are collinear.Prove that $OQ \perp EF \iff QE = QF$*



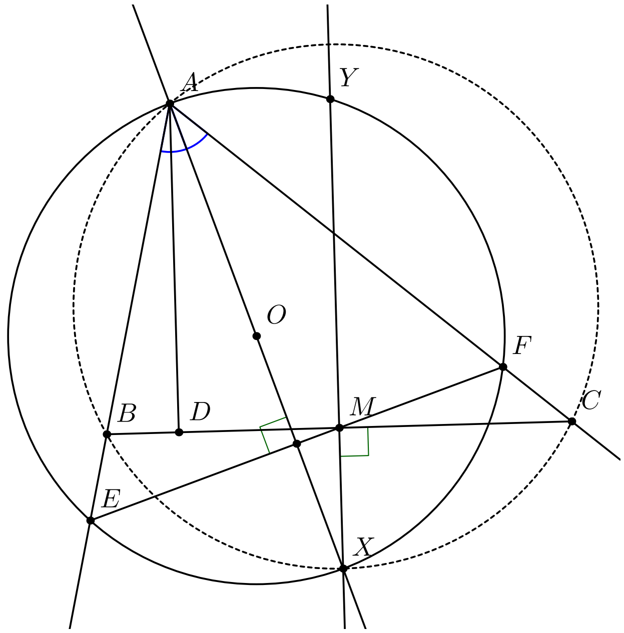
Solution. :This configuration should immediately remind you of the lemma above.First let us prove that $OQ \perp EF \Rightarrow EQ = EF$..So instantaneously we recognize that our attention should revolve around $AEOF$ to be cyclic,i.e $\angle BOF = 180 - A$.But this is a simple angle chase as $EOQB$ and $QOCF$ are cyclic.

Proving the other way round is a bit tricky.Let us see how our lemma helps us.Assume that $EQ = EF$.Let O' be on AM such that $O'Q \perp EF$.By our lemma,O' lies on circle ABC.Now drop perpendiculars B', C' from O' to AB, AC.By simson line,B', Q, C' must be collinear,and further since $AB' = AC'$ this cannot happen unless $B' = B, C' = C$.So the converse is done.

The only thing that is remaining is to 'formally' check the case when AEF is isosceles,which is easy. (Note that in subsequent problems we will omit such cases unless it is not trivial).

□

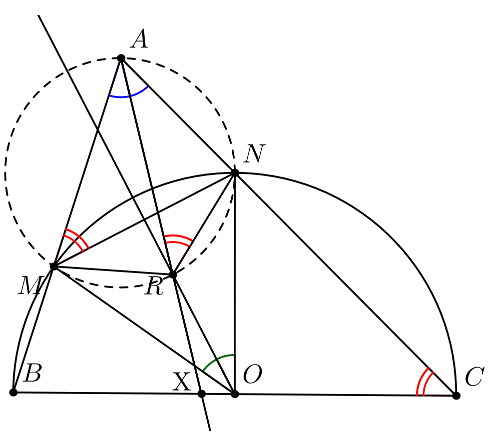
Problem 2. *:Given is a triangle ΔABC .The midpoint of BC is M.E, F lie respectively on AB, AC such that E, M, F are collinear and $AE = AF$.O is the centre of circumcircle of ΔAEF.If AD is altitude from A to BC,then prove that $OD = OM$*



Solution. Let the perpendicular to BC at M intersect AO at X.By lemma,X lies on $\odot ABC$.Now as we did in the previous problem,if we drop perpendiculars XE', XF' on AB, AC respectively,then by simson line,E', M, F' would be collinear.But XE' would be equal to XF' too,so $E = E' and F = F'$ is only possibility.Hence X lies on $\odot AFE$ too.

Now lets see if this is useful to solve the problem.Let XM intersect AEF at Y.As $\angle AYM = 90$,$AYMD$ is rectangle(Note that the main purpose of the above construction was to show this).And now we are done,since the radical axis of $\odot AEF, \odot AYMD$ bisects their radical axis AZ,and hence bisects DM too. □

Problem 3 (IMO 2004 P1)**.** *ABC is an acute angled triangle with AB.The circle with diameter BC intersects AB, AC respectively at M, N.O is the midpoint of BC.bisectors of $\angle BAC, \angle MON$ meet at R.Prove that $\odot BMR, \odot CNR$ meet again at point lying on BC*



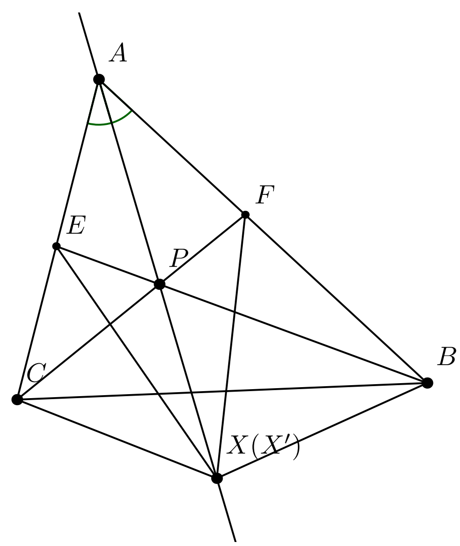
Solution. :We have that OR bisects MN.So by our lemma,$AMNR$ is cyclic.Now

$$\angle ACB = \angle AMN = \angle ARN$$

.This suggests that if AR is extended to meet BC at X then $CNRX$ is cyclic.Similarly we can show that $BXRM$ is cyclic and hence X is the required point lying on BC □

Now let us use our lemma to prove a generalization of the famous Steiner-Lehmus theorem.

Problem 4 (A generalisation of Steiner-Lehmus theorem)**.** *:ΔABC is a triangle.P is a point inside the triangle and lying on the angle bisector of $\angle BAC$.Cevians BP, CP meet opposite sides at E, F.Prove that $BE = CP \iff AB = AC$.*



Proof. :We observe that $\odot ACF, \odot AEB$ are congruent.Indeed this follows as both BE, CF are equal and subtend same angle in the circles $\odot ACF, \odot AEB$.We show that these two circles intersect again at X on line AP.Let $\odot ACF, \odot AEB$ intersect line AP X, X'.Since XA is angle bisector of $\angle BAC$,it follows that $\odot ACP$ is tangent to ACX.So we get

$$XP.XA = XC^2 = XF^2$$

.Similarly,

$$X'P.X'A = X'E^2 = X'B^2$$

But since $\odot ACF, \odot AEB$ are congruent,we have $XC = X'E$.This implies $XP.XA = X'P.X'A$ which is only possible if $X = X'$.So $XC = XB$ and our lemma implies X lies on $\odot ABC$ if $AB \neq AC$.But this gives a contradiction since it would imply A, E, C, X, B, F to be concyclic,which obviously cant be true.Hence $AB = AC$,as desired. □

4. Exercises

Here are some problems for the reader to try.Hints are given in the next section.

Exercise 1 (APMO 2000)**.** *:Let ΔABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA, respectively. And O the point in which the perpendicular at P to BA meets AN produced.*

Prove that QO is perpendicular to BC.

Exercise 2. *Given is a triangle ΔABC. M is on AB,N is on AC such that $BM = CN$. Prove that as M, N vary, $\odot AMN$ passes through a fixed point.*

Exercise 3 (IMOSL 2014 G3)**.** *Let and O be the circumcircle and the circumcentre of an acute angled triangle ΔABC with $AB > AC$. The angle bisector of $\angle ABC$ intersects at $M \neq B$. Let Γ be the circle with diameter BM. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect Γ at P and Q,respectively. R is on line PQ so that $BR = MR$. Prove that $BR \parallel AC$*

Exercise 4 (China TST 2015)**.** *$\triangle ABC$ is isosceles with $AB = AC > BC$. Let D be a point in its interior such that $DA = DB + DC$. Suppose that the perpendicular bisector of AB meets the external angle bisector of $\angle ADB$ at P, and let Q be the intersection of the perpendicular bisector of AC and the external angle bisector of $\angle ADC$. Prove that B, C, P, Q are concyclic.*

5. HINTS

1.Draw a line parallel to BC through O and exploit the configuration obtained.
2.The fixed point is the midpoint of arc BC containing A.
3.If X is midpoint of BM then $XOPQ$ is cyclic by lemma.
4.Apply the lemma and then sine rule chase.