Titu's Lemma

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Abstract

Titu's Lemma is a Lemma, discovered by *Titu Andreescu*, who was an USA IMO trainer. He found this result shortly after one of his lectures in *MOP 2001*, held at *Georgetown University* in the month of *June, 2001*. This particular Lemma has become very popular nowadays.

Titu's Lemma is actually a direct application of the *Cauchy-Schwarz* inequality, in short the *CS* inequality. This Lemma is also known as the *Engel Form* of the *CS* inequality.

1 The Lemma

Before stating the Lemma, let us recall the *CS* inequality, which says

Theorem 1 (The CS Inequality)**.** *For any real numbers* $a_1, a_2, \cdots, a_n$ *and* $b_1, b_2, \cdots, b_n$, *the following inequality holds.*

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$$

Equality occurs when either $a_k = 0$, *or* $b_k = 0$, *or* $a_k = b_k$, *or* $\dfrac{a_i}{b_i} = \dfrac{a_j}{b_j}$ $\forall\ i, j, k \in [1, 2, \cdots, n]$.

I will go through 2 proofs to this inequality.

First proof. Let us consider a quadratic polynomial

$$f(x) = \sum_{k=1}^{n}(a_kx - b_k)^2.$$

Now, we may write

$$f(x) = \sum_{k=1}^{n}(a_kx - b_k)^2 = \left(\sum_{k=1}^{n} a_k^2\right)x^2 - 2x\left(\sum_{k=1}^{n} a_kb_k\right) + \left(\sum_{k=1}^{n} b_k^2\right).$$

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The discriminant D of the polynomial $f(x)$ is equal to

$$\begin{aligned} D &= \left(2\sum_{k=1}^{n} a_k b_k\right)^2 - 4\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \\ &= 4\left[\left(\sum_{k=1}^{n} a_k b_k\right)^2 - \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)\right] \end{aligned}$$

Clearly, $f(x) \geq 0 \ \forall \ x \in \mathbb{R}$.

The polynomial $f(x)$ will vanish, if and only if $x = \dfrac{b_k}{a_k} \ \forall \ k \in [1, 2, \cdots, n]$.
So, either $f(x)$ will have 2 equal real roots, or it will have 2 non-real roots.
If $f(x)$ has 2 equal roots, then $f(x)$ will vanish.
That is, $x = \dfrac{b_k}{a_k} \ \forall \ k \in [1, 2, \cdots, n]$.
If $f(x)$ has non-real roots, then the discriminant $D \leq 0$.

Or, we have

$$\begin{aligned} 4\left[\left(\sum_{k=1}^{n} a_k b_k\right)^2 - \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)\right] &\leq 0 \\ \implies \left(\sum_{k=1}^{n} a_k b_k\right)^2 - \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) &\leq 0 \end{aligned}$$

And so

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) .$$

which proves the CS inequality.

$\square$

Second Proof. Let us try to use the *AM-GM* inequality, to prove the CS inequality. Let us denote

$$\left(\sum_{k=1}^{n} a_k^2\right) = S_a, \left(\sum_{k=1}^{n} b_k^2\right) = S_b.$$

By *AM-GM* inequality, we have

$$\frac{a_k^2}{S_a} + \frac{b_k^2}{S_b} \geq \frac{2a_k b_k}{\sqrt{S_a \cdot S_b}} \ \forall \ k \in [1, 2, \cdots, n].$$

Applying the above inequality gives us

$$\sum_{k=1}^{n}\left(\frac{a_k^2}{S_a}+\frac{b_k^2}{S_b}\right) \geq \sum_{k=1}^{n}\frac{2a_kb_k}{\sqrt{S_a\cdot S_a}}$$

$$\Longrightarrow \frac{2\left(\sum_{k=1}^{n} a_kb_k\right)}{\sqrt{S_a\cdot S_a}} \leq 2$$

$$\Longrightarrow \frac{\left(\sum_{k=1}^{n} a_kb_k\right)}{\sqrt{\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)}} \leq 1$$

$$\Longrightarrow \left(\sum_{k=1}^{n} a_kb_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right).$$

This proves the *CS* inequality.

□

There exists a generalization to the *CS* inequality, which is known as the *Hölder's* inequality. But I will not discuss that here. Instead, I will discuss that later, maybe in another article.

Now, let us state the *Titu's Lemma*. Which says

Theorem 2 (Titu's Lemma)**.** *For all real numbers $a_k, b_k\ \forall\ k \in [1,2,\ldots,n]$ such that $b_k \neq 0$, the following inequality holds.*

$$\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\cdots+\frac{a_n^2}{b_n} \geq \frac{(a_1+a_2+\cdots+a_n)^2}{b_1+b_2+\cdots+b_n}\ .$$

Equality occurs if and only if $\dfrac{a_i}{b_i} = \dfrac{a_j}{b_j}\ \forall\ i,j \in [1,2,\ldots,n]$.

First proof. Let us apply the *CS* inequality on 2 sets of reals, $\left[\dfrac{a_1}{\sqrt{b_1}}, \dfrac{a_2}{\sqrt{b_2}}, \ldots, \dfrac{a_n}{\sqrt{b_n}}\right]$ and $\left[\sqrt{b_1}, \sqrt{b_2}, \ldots, \sqrt{b_n}\ \right]$. We will get

$$\left(\sum_{k=1}^{n}\frac{a_k^2}{b_k}\right)\left(\sum_{k=1}^{n} b_k\right) \geq \left(\sum_{k=1}^{n}\frac{a_k}{\sqrt{b_k}}\cdot\sqrt{b_k}\right)^2 = \left(\sum_{k=1}^{n} a_k\right)^2.$$

$$\Longrightarrow \frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\cdots+\frac{a_n^2}{b_n} \geq \frac{(a_1+a_2+\cdots+a_n)^2}{b_1+b_2+\cdots+b_n}\ ,$$

which proves the *Titu's Lemma.*

□

Second proof. Let us induct on k.

For $k = 1$, the statement becomes $\frac{a^2}{b} \geq \frac{(a)^2}{b}$, which is obviously true.

For $k = 2$, the statement becomes $\frac{a^2}{c} + \frac{b^2}{d} \geq \frac{(a+b)^2}{c+d}$.

Cross-multiplication yields

$$\begin{aligned} \left(\frac{a^2}{c} + \frac{b^2}{d}\right)(c+d) &\geq (a+b)^2 \\ \implies a^2 + b^2 + \frac{a^2 d}{c} + \frac{b^2 c}{d} &\geq a^2 + 2ab + b^2 \\ \implies \frac{a^2 d}{c} + \frac{b^2 c}{d} &\geq 2ab. \qquad (1) \end{aligned}$$

which is obviously true by $AM - GM$, as $\frac{a^2 d}{c} + \frac{b^2 c}{d} \geq 2\sqrt{\frac{a^2 d}{c} \cdot \frac{b^2 c}{d}} = 2ab$.

Now, let us assume that for some positive integer k, the statement is true. That is, the following inequality is true.

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_k^2}{b_k} \geq \frac{(a_1 + a_2 + \cdots + a_k)^2}{b_1 + b_2 + \cdots + b_k}\ . \qquad (2)$$

Now by **(1)** and **(2)** we have

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_{k+1}^2}{b_{k+1}} \geq \frac{(a_1 + a_2 + \cdots + a_k)^2}{b_1 + b_2 + \cdots + b_k} + \frac{a_{k+1}^2}{b_{k+1}} \geq \frac{(a_1 + a_2 + \cdots + a_{k+1})^2}{b_1 + b_2 + \cdots + b_{k+1}}\ .$$

Thus by induction, we proved the *Titu's Lemma*.

□

Actually, we can prove the *CS* inequality using *Titu's Lemma*! That proof is also quite simple.

Third proof of CS Inequality. By *Titu's Lemma*, we have

$$\begin{aligned} a_1^2 + a_2^2 + \cdots + a_n^2 =& \frac{a_1^2 b_1^2}{b_1^2} + \frac{a_2^2 b_2^2}{b_2^2} + \cdots + \frac{a_n^2 b_n^2}{b_n^2} \\ \geq& \frac{(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2}{b_1^2 + b_2^2 + \cdots + b_n^2} \end{aligned}$$

$$\implies \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \geq \left(\sum_{k=1}^{n} a_k b_k\right)^2.$$

This proves the *CS* inequality.

□

2 Examples

As we have stated and proved *Titu's Lemma*, let's work on some problems using this result.

Problem 1 (Nesbitt's Inequality)**.** Let a, b, c be positive real numbers. Then prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} \ .$$

Solution. We can write $\dfrac{a}{b+c} = \dfrac{a^2}{ab+ca}$.

Similarly, $\dfrac{b}{c+a} = \dfrac{b^2}{bc+ab}$, $\dfrac{c}{a+b} = \dfrac{c^2}{ca+bc}$.

Adding the 3 inequalities gives us

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{a^2}{ab+ca} + \frac{b^2}{bc+ab} + \frac{c^2}{ca+bc} \ .$$

Now by *Titu's Lemma*, we get

$$\frac{a^2}{ab+ca} + \frac{b^2}{bc+ab} + \frac{c^2}{ca+bc} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)} \ .$$

Now, we know that

$$(a+b+c)^2 \geq 3(ab+bc+ca).$$

$$\implies \frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{3}{2} \ .$$

$$\begin{aligned} \implies \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &= \frac{a^2}{ab+ca} + \frac{b^2}{bc+ab} + \frac{c^2}{ca+bc} \\ &\geq \frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{3}{2} \\ \implies \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &\geq \frac{3}{2} \ . \end{aligned}$$

This completes the solution. □

Problem 2 (RMO 2013)**.** Let a, b, c, d, e be positive real numbers, each > 1. Then prove that the following inequality holds.

$$\frac{a^2}{c-1} + \frac{b^2}{d-1} + \frac{c^2}{e-1} + \frac{d^2}{a-1} + \frac{e^2}{b-1} \geq 20.$$

Solution. By applying *Titu's Lemma* on *LHS*, we get

$$\frac{a^2}{c-1} + \frac{b^2}{d-1} + \frac{c^2}{e-1} + \frac{d^2}{a-1} + \frac{e^2}{b-1} \geq \frac{(a+b+c+d+e)^2}{(a-1)+(b-1)+(c-1)+(d-1)+(e-1)} \ .$$

$$\implies \frac{a^2}{c-1} + \frac{b^2}{d-1} + \frac{c^2}{e-1} + \frac{d^2}{a-1} + \frac{e^2}{b-1} \geq \frac{(a+b+c+d+e)^2}{(a+b+c+d+e)-5} \ .$$

Let us define $S = a+b+c+d+e$. We get

$$\frac{a^2}{c-1} + \frac{b^2}{d-1} + \frac{c^2}{e-1} + \frac{d^2}{a-1} + \frac{e^2}{b-1} \ge \frac{S^2}{S-5}\ .$$

Thus, it remains to prove that

$$\begin{aligned} \frac{S^2}{S-5} &\ge 20. \\ \implies S^2 &\ge 20S - 100. \\ \implies S^2 - 20S + 100 &\ge 0. \\ \implies (S-10)^2 &\ge 0, \text{ which is obvious.} \end{aligned}$$

This completes the proof. □

Problem 3 (Croatia 2004, RMO 2006, Moscow 2008)**.** Let a, b, c be positive real numbers. Then prove that

$$\frac{a^2}{(a+b)(a+c)} + \frac{b^2}{(a+b)(b+c)} + \frac{c^2}{(c+a)(c+b)} \ge \frac{3}{4}\ .$$

Solution. By *Titu's Lemma*, we get

$$\sum_{\text{cyc}} \frac{a^2}{(a+b)(a+c)} \ge \frac{\left(\sum_{\text{cyc}} a\right)^2}{\sum_{\text{cyc}} a^2 + 3\sum_{\text{cyc}} ab} = \frac{\sum_{\text{cyc}} a^2 + 2\sum_{\text{cyc}} ab}{\sum_{\text{cyc}} a^2 + 3\sum_{\text{cyc}} ab}\ .$$

So, it remains to prove that

$$\frac{\sum_{\text{cyc}} a^2 + 2\sum_{\text{cyc}} ab}{\sum_{\text{cyc}} a^2 + 3\sum_{\text{cyc}} ab} \ge \frac{3}{4}\ .$$

This is equivalent to proving

$$\begin{aligned} 4\sum_{\text{cyc}} a^2 + 8\sum_{\text{cyc}} ab &\ge 3\sum_{\text{cyc}} a^2 + 9\sum_{\text{cyc}} ab. \\ \iff \sum_{\text{cyc}} a^2 &\ge \sum_{\text{cyc}} ab, \text{ which is obvious.} \end{aligned}$$

This completes the proof. □

Problem 4 (IMO 1995)**.** Let a, b, c be positive real numbers with product 1. Then prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \ge \frac{3}{2}\ .$$

Solution. Let us substitute $a = \dfrac{1}{x}$, $b = \dfrac{1}{y}$, $c = \dfrac{1}{z}$.
As $abc = 1, xyz = 1$.
We get

$$\begin{aligned}\sum_{\text{cyc}} \frac{1}{a^3(b+c)} &= \sum_{\text{cyc}} \frac{1}{\frac{1}{x^3}\cdot\left(\frac{1}{y}+\frac{1}{z}\right)}\\ &= \sum_{\text{cyc}} \frac{1}{\left(\frac{y+z}{x^3yz}\right)}\\ &= \sum_{\text{cyc}} \frac{x^2}{y+z} \ .\end{aligned}$$

By *Titu's Lemma* and *AM-GM*, we get

$$\sum_{\text{cyc}} \frac{1}{a^3(b+c)} = \sum_{\text{cyc}} \frac{x^2}{y+z} \geq \frac{(x+y+z)^2}{2(x+y+z)} = \frac{x+y+z}{2} \geq \frac{3\sqrt[3]{xyz}}{2} = \frac{3}{2} \ .$$

Hence we get

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2} \ .$$

This completes the proof. $\square$

Problem 5 (Eeshan Banerjee)**.** Let a, b, c be positive real numbers such that $abc = 1$. Then prove that $\displaystyle\sum_{\text{cyc}} \frac{a^3}{b+c} \geq \frac{3}{2}$.

Solution. We may write $\displaystyle\sum_{\text{cyc}} \frac{a^3}{b+c} = \sum_{\text{cyc}} \frac{a^4}{ab+ac}$.
Now by *Titu's Lemma*, we get

$$\begin{aligned}\sum_{\text{cyc}} \frac{a^4}{ab+ac} &\geq \frac{\left(a^2+b^2+c^2\right)^2}{2(ab+bc+ca)}\\ \implies \sum_{\text{cyc}} \frac{a^3}{b+c} &\geq \frac{\left(a^2+b^2+c^2\right)^2}{2(ab+bc+ca)}\\ &\geq \frac{\left(a^2+b^2+c^2\right)^2}{2\left(a^2+b^2+c^2\right)}\\ &= \frac{\left(a^2+b^2+c^2\right)}{2}\end{aligned}$$

$$
\begin{aligned}
\implies \sum_{\text{cyc}} \frac{a^3}{b+c} &= \frac{\left(a^2+b^2+c^2\right)}{2} \\
\implies \sum_{\text{cyc}} \frac{a^3}{b+c} &\geq \left(\frac{a+b+c}{3}\right)^2 \cdot \frac{3}{2} && \text{(Power mean)} \\
&\geq \left(\frac{3\sqrt[3]{abc}}{3}\right)^2 \cdot \frac{3}{2} \\
&= \left(\frac{3}{3}\right)^2 \cdot \frac{3}{2} = \frac{3}{2} && (\text{AM} - \text{GM},\ abc = 1) \\
\implies \sum_{\text{cyc}} \frac{a^3}{b+c} &\geq \frac{3}{2}\ .
\end{aligned}
$$

This completes the proof. □

Problem 6. For positive reals a, b, c, prove the inequality

$$\frac{9}{a+b+c} \leq \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\ .$$

Solution. This inequality can be easily proven using *Titu's Lemma.*

Proof For Left Inequality. By *Titu's Lemma*, we get

$$\sum_{\text{cyc}} \frac{2}{a+b} \geq \frac{\left(\sqrt{2}+\sqrt{2}+\sqrt{2}\right)^2}{\sum_{\text{cyc}}(a+b)} = \frac{9}{a+b+c}\ .$$

This proves the Left Inequality.

Proof For Right Inequality. Again by *Titu's Lemma*, we get

$$\sum_{\text{cyc}} \frac{1}{a} = \frac{\sum_{\text{cyc}} \left(\frac{1}{a} + \frac{1}{b}\right)}{2} \geq \frac{\sum_{\text{cyc}} \frac{(1+1)^2}{a+b}}{2} = \sum_{\text{cyc}} \frac{2}{a+b}.$$

This proves the right inequality. □

Problem 7. Let $a_1, a_2, \ldots, a_n$ be positive reals. Let $s = a_1 + a_2 + \cdots + a_n$. Then prove that

$$\sum_{k=1}^{n} \frac{a_k}{s - a_k} \geq \frac{n}{n-1}\ .$$

Solution. We can write $\displaystyle\sum_{k=1}^{n} \frac{a_k}{s-a_k} = \sum_{k=1}^{n} \frac{a_k^2}{sa_k - a_k^2}\ .$

Now by *Titu's Lemma*, we get

$$\begin{aligned}
\sum_{k=1}^{n} \frac{a_k^2}{sa_k - a_k^2} &\geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{s(a_1 + a_2 + \cdots + a_n) - (a_1^2 + a_2^2 + \cdots + a_n^2)} \\
&\geq \frac{s^2}{s^2 - n \cdot \left(\dfrac{a_1 + a_2 + \cdots + a_n}{n} \right)^2} && \text{(Power Mean)} \\
&= \frac{s^2}{s^2 - \dfrac{s^2}{n}} = \frac{n}{n-1} \ .
\end{aligned}$$

This completes the proof. □

Problem 8. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. And let s be the sum of them. That is, let $s = x_1 + x_2 + \cdots + x_n$. Then prove that

$$\sum_{k=1}^{n} \frac{s}{s - x_k} \geq \frac{n^2}{n-1} \ .$$

Solution. We may write $\displaystyle\sum_{k=1}^{n} \frac{s}{s - x_k} = s \left(\sum_{k=1}^{n} \frac{1}{s - x_k} \right)$.
And by *Titu's Lemma*, we get

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{s - x_k} &\geq \frac{(1 \cdot n)^n}{ns - (x_1 + x_2 + \cdots + x_n)} \\
&= \frac{n^2}{ns - s} = \frac{1}{s} \cdot \left(\frac{n^2}{n-1} \right) . \\
\implies s \cdot \left(\sum_{k=1}^{n} \frac{1}{s - x_k} \right) &\geq \frac{n^2}{n-1} \\
\implies \sum_{k=1}^{n} \frac{s}{s - x_k} &\geq \frac{n^2}{n-1} \ .
\end{aligned}$$

This completes the solution. □

Problem 9. Let a, b, c be sides of a triangle. Prove that $\displaystyle\sum_{\text{cyc}} \frac{a}{b + c - a} \geq 3$.

Solution. We may write $\displaystyle\sum_{\text{cyc}} \frac{a}{b + c - a} = \sum_{\text{cyc}} \frac{a}{ab + ac - a^2} \ .$

Now by *Titu's Lemma*, we get

$$\begin{aligned}
\sum_{\text{cyc}} \frac{a^2}{ab+ac-a^2} &\geq \sum_{\text{cyc}} \frac{(a+b+c)^2}{2(ab+bc+ca)-(a^2+b^2+c^2)} \\
&\geq \frac{(a+b+c)^2}{ab+bc+ca} && \left[\because a^2+b^2+c^2 \geq ab+bc+ca\right] \\
&\geq \frac{3(ab+bc+ca)}{ab+bc+ca} = 3 && \left[\because (a+b+c)^2 \geq 3(ab+bc+ca)\right] \\
\implies \sum_{\text{cyc}} \frac{a}{b+c-a} &\geq 3.
\end{aligned}$$

This completes the proof. $\square$

Problem 10. Let a, b, and c be real numbers. Prove that

$$2a^2+3b^2+6c^2 \geq (a+b+c)^2.$$

Solution. Let us rewrite the $LHS = 2a^2+3b^2+6c^2 = \dfrac{a^2}{1/2} + \dfrac{b^2}{1/3} + \dfrac{c^2}{1/6}$.
Then by *Titu's Lemma*, we get

$$\text{LHS} \geq \frac{(a+b+c)^2}{1/2+1/3+1/6} = (a+b+c)^2.$$

This completes the proof. $\square$

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