A Taste Of Inversion

Adithya Bhaskar, Anant Mudgal

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1 Introduction

Here we give a simple definition of inversion and a few other definitions which shall be useful for this article.

Definition 1. An *inversion* with center O and radius R transforms every point P in space to the point P', where P' is defined by the following conditions:

(a) $P' \in OP$

(b) $OP \cdot OP' = R^2$.

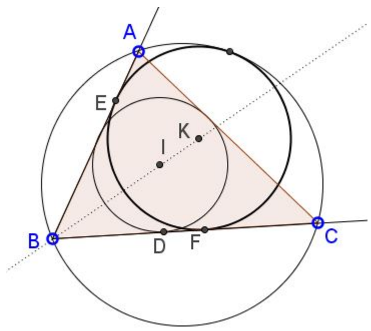
Definition 2. Consider a reference triangle $\triangle ABC$ and a point P. Let the reflection of P in the internal angle bisectors of $\angle A, \angle B, \angle C$ be denoted by P_a, P_b, P_c respectively. Then AP_a, BP_b and CP_c are concurrent at a point P' which is said to be the *isogonal conjugate* of P w.r.t $\triangle ABC$.

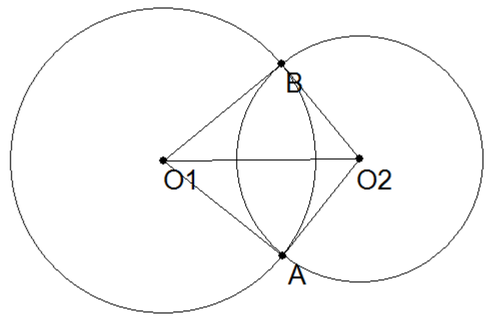
Definition 3. For a point P in the plane of $\triangle ABC$, let us define $f(P) = |AP| + |BP| + |CP|$. Then the *First Fermat Point* of the triangle $\triangle ABC$ is defined to be the point P such that $f(P)$ is minimum.

Warmup 1. Prove that for the Fermat point P, we have $\angle APB = \angle BPC = \angle CPA$. (Hint : Rotate.)

Definition 4. Consider a triangle $\triangle ABC$ and its circumcircle Γ. Then the A-mixtilinear incircle Γ_A is defined to be the cirle tangent to AB, AC and internally tangent to Γ.

Definition 5. Let us consider two circles $\odot(O_1), \odot(O_2)$ intersecting in A, B. The circles are said to be orthogonal iff $O_1A \perp AO_2, O_1B \perp O_2B$.





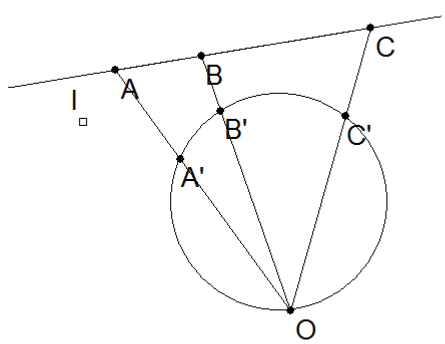
2 Basic results

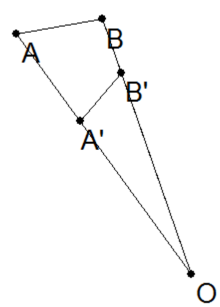
Result 1. The image of a line under inversion is a circle which passes through the point about which the inversion is performed, O. In particular, if the line passes through through O then its image is a circle of infinite radius.

Result 2. Let ω be a circle through A and let ω' be its image under an arbitrary inversion with center A. Then ω' is a line.

Result 3. Let A, B two points such that A, O, B are not collinear. Then $\triangle OAB \sim \triangle OB'A'$. (Throughout the article X' shall denote the image of X under the inversion; except when specified otherwise).

Result 4. Let ω_1, ω_2 be two orthogonal circles. Then the inverse of ω_2 w.r.t ω_1 is ω_2 itself.





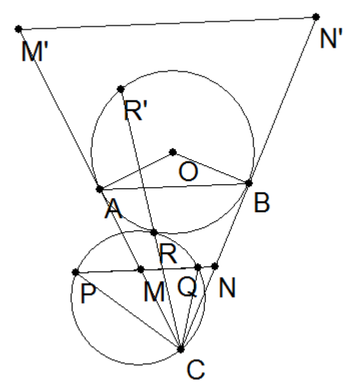
3 Examples.

So here we include a wide range of examples for our reader. In the forthcoming articles we shall show the reader different *types* of Inversion, but at this stage we confine ourselves to the basic definition. These examples are quite hard so the reader is advised to spend enough time on each one. And don't worry if you don't understand everything in one go, it takes a lot of experience to get accustomed to inversion!

So let us start with a problem that was a part of Korea 2014.

1. (Korea 2014/3) AB is a chord of O and AB is not a diameter of O. The tangent lines to O at A and B meet at C. Let M and N be the midpoint of the segments AC and BC, respectively. A circle passing through C and tangent to O meets line MN at P and Q. Prove that $\angle PCQ = \angle CAB$.

Solution. This solution is due to Telv Cohl.



Let M', N' be the reflection of C in A, B, respectively. Invert with center C and factor $CA^2 = CB^2$. Since M', N' are the images of M, N under this inversion, respectively so the image of MN under this inversion is $(CM'N')$ which is a circle concentric with (O) . ... $(\star)$

Since the image P', Q' of P, Q under this inversion lie on $(CM'N')$ and $P'Q'$ is tangent to (O) . so from $(\star) \Longrightarrow P'Q' = CM' = CN' \Longrightarrow \angle PCQ = \angle P'CQ' = \angle N'M'C = \angle BAC$.

In the above solution, note that $CM' = 2CM$ etc. that's why OM is the perpendicular bisector of $CM' \Longrightarrow O$ is the circumcenter of $CM'N'$ easily.

We proceed with an example that turned up in the IMO Shortlist 2014.

2. (IMO Shortlist 2014) Consider a fixed circle Γ with three fixed points A, B, and C on it. Also, let us fix a real number $\lambda \in (0,1)$. For a variable point $P \notin \{A, B, C\}$ on Γ, let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC. Prove that as P varies, the point Q lies on a fixed circle.

Solution. This solution is attributed to Evan Chen. It is noted that a proof is also possible along the lines of the Miquel Point of a complete quaderilateral.

Do an inversion around C. Then $\triangle CAB$ is fixed after inversion, and M moves on a line parallel to AB; $P = AB \cap CM$, and Q is the intersection of BM with the circumcircle of APC.

By Power of a Point (in directed form),

$$BM \cdot BQ = BP \cdot BA$$

so Q is the point on the line BM such that $BQ = c \cdot \frac{BP}{BM}$, where $c = BA$.

Now, let R be the point on the line BM such that $BR = c \cdot \frac{BM}{BP}$. We claim that R moves along a line. Indeed, let $C = (0,0)$, $P = (p,1)$, $B = (b,1)$, $M = (hp,h)$, then $M - B = (hp - b, h - 1)$ so

$$R = B + \left(\frac{hp - b}{b - p}, \frac{h - 1}{b - p}\right)$$

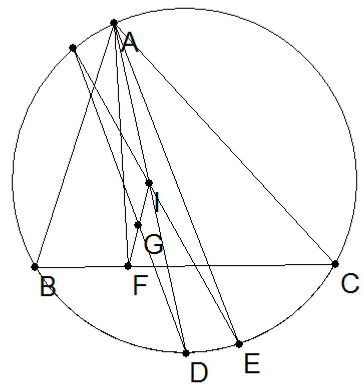
so that $R - B$ satisfies $-x + by = h$, which is a line.

Now, R is the image of Q under an inversion at B of radius c, so Q moves on a circle.

We continue with a problem proposed at the IMO 2010.

3. (IMO 2010) Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < 12\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ.

Solution.



This solution is by Sung-Yoon Kim:

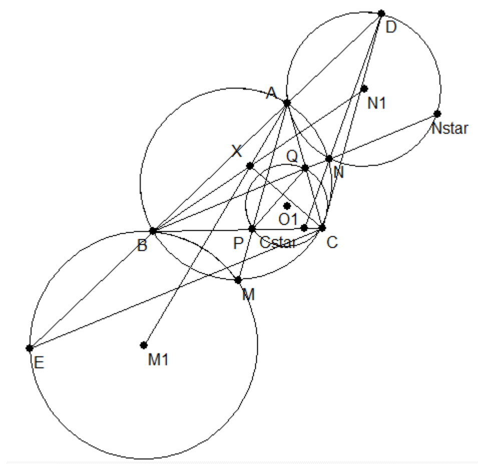
We show that $\triangle AFI_a$ and $\triangle AIE$ are similar. Then we have $\angle AEI = \angle AI_aF = \angle ADG$ and we're done. To show that, invert the plane with regard to A with the radius $\sqrt{bc}$ where $b = AC, c = AB$. Then we have another figure which can be also obtained by reflecting the original figure. Note that E, F are mapped to F, E resp. Hence $AE \cdot AF = bc = AI \cdot AI_a$, which implies directly that $\triangle AFI_a$ and $\triangle AIE$ are similar, as desired.

Here, we wish to make three remarks. First, the figure just given for visualization sake: the problem solution's points are not in the figure. Second, on the mathlinks forum, many users had posted solutions using projective geometry. It is generally noted that projective geometry and inversion are brother and sister. So if inversion makes the problem easier but not trivial, try projective geometry after inverting! Thirdly, note that if we define Ψ to be the composition of an inversion about A with radius $\sqrt{AB \cdot AC}$ and a reflection about AI, then $\Psi(ABC) = \Psi(ACB)$. This property is often used (as above) because Ψ preserves the reference triangle. This sometimes turns out to be a crucial point in solving problems.

Now the level of the example problems ascends: we go for a harder problem proposed by mihajlon of artofproblemsolving, then look a a few problems which are even harder.

4. (Mihajlon) Triangle $\triangle ABC$ is given. Points D i E are on line AB such that $D - A - B - E, AD = AC$ and $BE = BC$. Bisector of internal angles at A and B intersect BC, AC at P and Q, and circumcircle of ABC at M and N. Line which connects A with center of circumcircle of BME and line which connects B and center of circumcircle of AND intersect at X. Prove that $CX \perp PQ$.

Solution. This solution is due to Telv Cohl.



Let M_1, N_1 be the center of $\odot(BME), \odot(AND)$, respectively . Let O_1 be the center of $\odot(CPQ)$ and $N^* = BN \cap \odot(N_1), C^* = CB \cap DN^*$.

Since $BA \cdot BD = BN \cdot BN^*$, so $A \longleftrightarrow D, N \longleftrightarrow N^*$ under inversion $\mathbf{I}(B, \sqrt{BA \cdot BD}) = \Psi$, hence $\odot(ABC) \longleftrightarrow DN^*$ under $\Psi \Longrightarrow C^*$ is the image of C under Ψ .

Since $BA : BP = (BA + AC) : BC = BD : BC$, so from $\triangle BAC \cup Q \sim \triangle BC^*D \cup N^* \Longrightarrow \triangle BAC \cup Q \cup P \cup O_1 \sim \triangle BC^*D \cup N^* \cup A \cup N_1$, hence we get $\triangle BCO_1 \sim \triangle BDN_1 \Longrightarrow BO_1, BN_1$ are isogonal conjugate of $\angle ABC$.

Similarly we can prove AO_1, AM_1 are isogonal conjugate of $\angle CAB$, so O_1 is the isogonal conjugate of X WRT $\triangle ABC \Longrightarrow CX \perp PQ$.

If the reader is acquainted with the concept of harmonic pencils then hey may try this next example, proposed by Tran Quang Hung:

5. (Tran Quang Hung) Let ABC be a triangle with first Fermat point F. Let A_b, A_c lie on BC such that triangle FA_bA_c is equilateral with A_b is between B and A_c. Similarly, we have points B_c, B_a, C_a, C_b. Prove that circumcircles of triangles $FB_cC_b, FC_aA_c, FA_bB_a$ are coaxal.

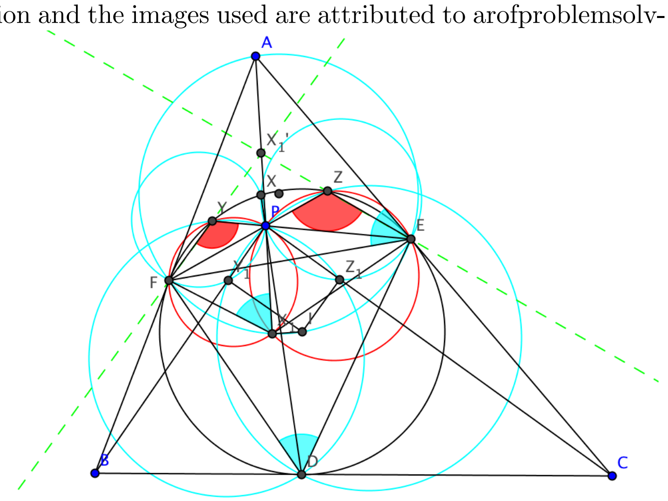
Solution. This solution is by Telv Cohl. Let $\triangle F_aF_bF_c$ be the pedal triangle of F WRT $\triangle ABC$. Since $\triangle A_bB_cC_a$, $\triangle A_cB_aC_b$ is the image of $\triangle F_aF_bF_c$ under the spiral similarity $\mathbf{S}(F, -30°, 2\sqrt{3})$, $\mathbf{S}(F, 30°, 2\sqrt{3})$, respectively, so $\triangle A_bB_cC_a \cup F \sim \triangle F_aF_bF_c \cup F \sim \triangle A_cB_aC_b \cup F \Longrightarrow F$ is the first Isodynamic point of $\triangle A_bB_cC_a$ and $\triangle A_cB_aC_b$. Performing the inversion with center F and let V^* be the image of V. Since $\triangle A_b^*B_c^*C_a^*$, $\triangle A_c^*B_a^*C_b^*$ are equilateral triangles and $\triangle A_c^*B_a^*C_b^*$ is the image of $\triangle A_b^*B_c^*C_a^*$ under the rotation $\mathbf{R}(F, 60°)$, so

$\triangle A_b^* B_c^* C_a^*$ and $\triangle B_a^* C_b^* A_c^*$ are homothetic (and congruent) $\Longrightarrow$ $B_c^* C_b^*$, $C_a^* A_c^*$, $A_b^* B_a^*$ are concurrent, hence we conclude that $\odot(FB_cC_b)$, $\odot(FC_aA_c)$, $\odot(FA_bB_a)$ are coaxial.

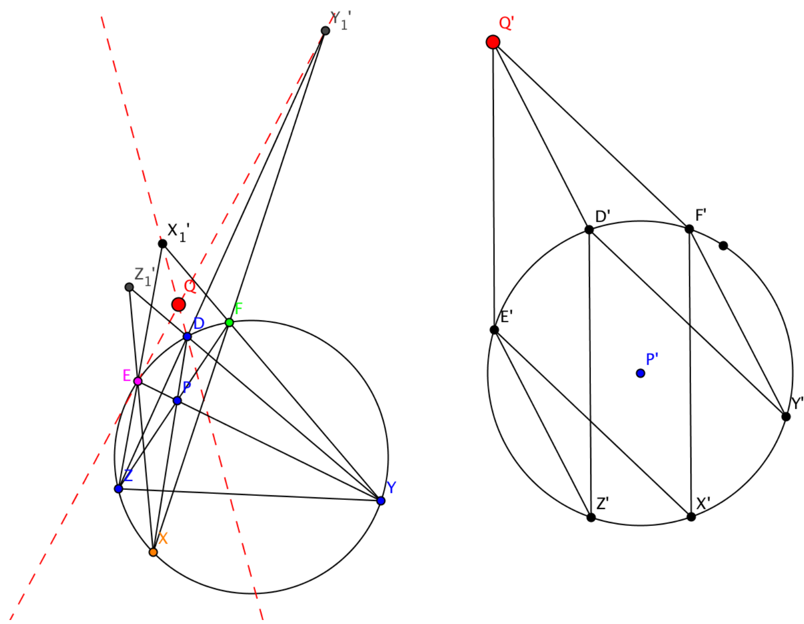
Lastly, we go for a problem proposed by artofproblemsolving user SalaF:

6. (SalaF) Let ABC be a triangle such that its incircle $I(\gamma)$ touches BC, CA, AB at D, E, F. Let P a point in the interior of γ and let DP, EP, CP meet again γ at X, Y, Z. Let X_1, Y_1, Z_1 be the projections of I on the lines AP, BP, CP. Show that the circles $\odot(PXX_1), \odot(PYY_1), \odot(PZZ_1)$ have a second common point (they are coaxal).

Solution. This solution and the images used are attributed to arofproblemsolv-



ing user mineiraojose.



As X_1 lies on $(AI) \Rightarrow \angle PX_1F = \angle AEF = \angle EDF$, but $\angle FYP = 180° - \angle EDF$ so F, Y, P, X_1 are concyclic. Analogously $X_1 \in \odot(PZE)$.

So redefine $X_1 = \odot(PFY) \cap \odot(PZE)$, $Y_1 = \odot(PDZ) \cap \odot(PXF)$, $Z_1 = \odot(PYD) \cap \odot(PXE)$.

Denote by $\Psi(X)$ the inverse of X under inversion $(P, \sqrt{-PX \cdot PD})$. Obviously $X \longleftrightarrow D$, $Y \longleftrightarrow E$, $F \longleftrightarrow Z$ so $\Psi(X_1) = EZ \cap FY$, $\Psi(Y_1) = DZ \cap FX$, $\Psi(Z_1) = YD \cap XE$.

Now the problem is to prove that for X, Z, E, D, F, Y on the circle ω with $\Psi(X_1) = EZ \cap FY$, $\Psi(Y_1) = DZ \cap FX$, $\Psi(Z_1) = YD \cap XE$, that $\Psi(Z_1)F, \Psi(X_1)D, \Psi(Y_1)E$ are concurrent at a point Q. But considering the projective transformation that sends P into the center of ω, it's easy to see that Q is the intersection of the parallel through E to $DZ||FX$, parallel through F to to $DY||EX$ and the parallel through D to $EZ||FY$. So$\odot(PXX_1), \odot(PYY_1), \odot(PZZ_1)$ are coaxal. In exactly the same way, we can prove that $\odot(PDX_1), \odot(PEY_1), \odot(PFZ_1)$ are also coaxal.

Okay, I know what some of the readers are feeling right now. Isn't this too much at this level? Yes, but looking at these solutions will teach you and me how professionals use inversion. Since the 'beginner' public must not feel left out, I am including a section before the problem section.

4 When to use inversion, and how?

Okay, that's a quesion with a pretty long answer.

When to use inversion? Well, some part of it is learnt through intuition, but let's see what inversion basically does. It looks like some wierd power of point; and that's what it does: transforms circles to better circles (note that a line is a circle of infinite radius). It enables the use of power of point. As we shall see in the forthcoming articles, there are different types of inversion for different problems. At this point I think an appropriate answer is "When there are too many circles, or too many circles (which include lines) passing through a point, inversion is a good option".

How? That's what you saw in the above section. However, we shall give you a slight glimpse into what's coming the future versions of this article:

- **Common Inversion.** This is what one can call : Invert just based on intuition,
- $\sqrt{bc}$ You must have observed above solutions which said "Ψ is a composition of an inversion with radius $\sqrt{AB\dot{A}C}$ and a reflection about the angle bisector of $\angle A$. Why so? Note that this inversion preserves the triangle $\triangle ABC$. So when you don't want your base triangle to change (it's pretty nice) then this is a good idea.
- **Orthogonal** Example 1 is this. They have a nice property of inverting into themselves, so this turns out to be quite handy.
- **Incentral.**
- **Orthic Inversion.**
- **Overlaying.**

And a few more.... so now on to problems.

5 Problems! :D

1. (IMO Shortlist 2002) Let B be a point on a circle S_1, and let A be a point distinct from B on the tangent at B to S_1. Let C be a point not on S_1 such that the line segment AC meets S_1 at two distinct points. Let S_2 be the circle touching AC at C and touching S_1 at a point D on the opposite side of AC from B. Prove that the circumcentre of triangle BCD lies on the circumcircle of triangle ABC.

2. Let ABC be a triangle, and let the tangent to the circumcircle of the triangle ABC at A meet the line BC at D. The perpendicular to BC at B meets the perpendicular bisector of AB at E. The perpendicular to BC at C meets the perpendicular bisector of AC at F. Prove that the points D, E and F are collinear.

3. (livetolove2) Let ABC be a triangle which $AB + AC = 3BC$. I is incenter of the triangle. (I) is tangent to BC, CA, AB at D, E, F, resp. K, L is symmetric of E, F through I, resp. The circle with diameter AI intersect (O) again at T. Prove that (LKT) is tangent to (O).

4. (andria) In $\triangle ABC$ let the A-mixtilinear incircle cut BC at $R, T(R$ between $B, T)$ ω is a circle that it passes throw A and it is tangent to A-mixtilinear incircle and BC at E, F prove that $\odot(\triangle AFT)$ is tangent to the A-mixtilinear incircle at T.

5. (Kapil Pause) The incircle ω of triangle ABC touches BC at D. Let $P \in \omega$ such that (BPC) is tangent to ω at P. Let AP meet (BPC) again at Q and I be the center of ω. Prove that QI bisects the mid-point of arc BC of (BPC) not containing Q.

6. (Kapil Pause again :D) In triangle ABC, let O be the circumcenters and M be the midpoint of BC. Let T be a point where the circle through the midpoints of AB, AC touches the circumcircle of ABC and T is not A. Let AT meet the circle through the midpoints of AB, AC and T again at X. Prove that T, X, O, M are con cyclic.

7. (d990528) Let $\odot(O)$ be the circumscribed circle of triangle ABC. Let W_a be the A-Mixtilinear incircle and W_a meet $\odot(O)$ in D, AB in P, AC in Q. Define X as the intersection of PQ and AD. Prove that $\angle BXP = \angle CXQ$.

8. (In The World Of Mathematics/482) Let ω be the circumcircle of ABC, l be the tangent line to the circle ω at point A. The circles ω_1 and ω_2 touch lines l, BC and the circle ω externally. Denote by D, E the points where ω_1, ω_2 touch BC. Prove that the circumcircles of triangles ABC and ADE are tangent.

9. (China TST 2014) Let the circumcenter of triangle ABC be O. H_A is the projection of A onto BC. The extension of AO intersects the circumcircle of BOC at A'. The projections of A' onto AB, AC are D, E, and O_A is the circumcentre of triangle DH_AE. Define H_B, O_B, H_C, O_C similarly. Prove that H_AO_A, H_BO_B, H_CO_C are concurrent.

10. (Geolympiad 2015 Fall) In triangle ABC, let Ω_a be the A Mixtilinear Incircle. Let P_A be a point on it such that the circumcircle of BP_AC is tangent to it at P_A. Similarly, define P_B, P_C. Prove that AP_A, BP_B, CP_C are all concurrent.

11. (drmzjoseph, hard) Let F the foot of the altitude from C of the $\triangle ABC$. Let ω_1 and ω_2 the incircles of $\triangle ACF$ and $\triangle BCF$ respectively, the common external tangent, distinct of AB, to ω_1 and ω_2 touch ω_1 and ω_2 at X and Y respectively.$XY \cap CB \equiv M; XY \cap AC \equiv N : AX \cap BY = P; AM \cap BN = Q$. Prove that C, P and Q are collinear.

12. (Tran Quang Hung, hard) Let ABC be a triangle inscribed in circle (O). Construct inside two similar triangles $\triangle BAM \sim \triangle CAN$. Construct outside triangles $\triangle BKA \sim \triangle BAM$ and $\triangle CLA \sim \triangle CAN$. Let U, V be circumcenters of triangles AMN và AKL. Q is midpoint of BC. Prove that pencil $O(UV, AQ)$ is harmonic.

6 References

[1] *Various posts at Artofproblemsolving*
http://www.artofproblemsolving.com