Algebraic Inequalities in Mathematical Olympiads: Problems and Solutions

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Abstract

This is a collection of recent algebraic inequalities proposed in math Olympiads from around the world.

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1 Problems

1. (Azerbaijan JBMO TST 2015) With the conditions $a, b, c \in \mathbf{R}^+$ and $a + b + c = 1$, prove that

$$\frac{7+2b}{1+a} + \frac{7+2c}{1+b} + \frac{7+2a}{1+c} \geq \frac{69}{4}$$

2. (Azerbaijan JBMO TST 2015) $a, b, c \in \mathbf{R}^+$ and $a^2 + b^2 + c^2 = 48$. Prove that

$$a^2\sqrt{2b^3+16} + b^2\sqrt{2c^3+16} + c^2\sqrt{2a^3+16} \leq 24^2$$

3. (Azerbaijan JBMO TST 2015) $a, b, c \in \mathbf{R}^+$ prove that

$$[(3a^2+1)^2+2(1+\frac{3}{b})^2][(3b^2+1)^2+2(1+\frac{3}{c})^2][(3c^2+1)^2+2(1+\frac{3}{a})^2] \geq 48^3$$

4. (AKMO 2015) Let a, b, c be positive real numbers such that $abc = 1$. Prove the following inequality:

$$a^3 + b^3 + c^3 + \frac{ab}{a^2+b^2} + \frac{bc}{b^2+c^2} + \frac{ca}{c^2+a^2} \geq \frac{9}{2}$$

5. (Balkan MO 2015) If a, b and c are positive real numbers, prove that

$$a^3b^6 + b^3c^6 + c^3a^6 + 3a^3b^3c^3 \geq abc\left(a^3b^3 + b^3c^3 + c^3a^3\right) + a^2b^2c^2\left(a^3 + b^3 + c^3\right).$$

6. (Bosnia Herzegovina TST 2015) Determine minimum value of the following expression:

$$\frac{a+1}{a(a+2)} + \frac{b+1}{b(b+2)} + \frac{c+1}{c(c+2)}$$

for positive real numbers such that $a + b + c \leq 3$

7. (China 2015) Let $z_1, z_2, ..., z_n$ be complex numbers satisfying $|z_i - 1| \le r$ for some $r \in (0,1)$. Show that

$$\left|\sum_{i=1}^{n} z_i\right| \cdot \left|\sum_{i=1}^{n} \frac{1}{z_i}\right| \ge n^2(1-r^2).$$

8. (China TST 2015) Let $a_1, a_2, a_3, \cdots, a_n$ be positive real numbers. For the integers $n \ge 2$, prove that

$$\left(\frac{\sum_{j=1}^{n}\left(\prod_{k=1}^{j} a_k\right)^{\frac{1}{j}}}{\sum_{j=1}^{n} a_j}\right)^{\frac{1}{n}} + \frac{\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}}{\sum_{j=1}^{n}\left(\prod_{k=1}^{j} a_k\right)^{\frac{1}{j}}} \le \frac{n+1}{n}$$

9. (China TST 2015) Let $x_1, x_2, \cdots, x_n$ $(n \ge 2)$ be a non-decreasing monotonous sequence of positive numbers such that $x_1, \frac{x_2}{2}, \cdots, \frac{x_n}{n}$ is a non-increasing monotonous sequence .Prove that

$$\frac{\sum_{i=1}^{n} x_i}{n\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}} \le \frac{n+1}{2\sqrt[n]{n!}}$$

10. (Junior Balkan 2015) Let a, b, c be positive real numbers such that $a+b+c=3$. Find the minimum value of the expression

$$A = \frac{2-a^3}{a} + \frac{2-b^3}{b} + \frac{2-c^3}{c}.$$

11. (Romania JBMO TST 2015) Let x,y,$z > 0$. Show that :

$$\frac{x^3}{z^3+x^2y} + \frac{y^3}{x^3+y^2z} + \frac{z^3}{y^3+z^2x} \ge \frac{3}{2}$$

12. (Romania JBMO TST 2015) Let $a, b, c > 0$ such that $a \ge bc^2$, $b \ge ca^2$ and $c \ge ab^2$. Find the maximum value that the expression :

$$E = abc(a-bc^2)(b-ca^2)(c-ab^2)$$

can acheive.

13. (Romania JBMO TST 2015) Prove that if $a, b, c > 0$ and $a+b+c=1$, then

$$\frac{bc+a+1}{a^2+1} + \frac{ca+b+1}{b^2+1} + \frac{ab+c+1}{c^2+1} \le \frac{39}{10}$$

14. (Kazakhstan 2015) Prove that

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n+1)^2} < n \cdot \left(1 - \frac{1}{\sqrt[n]{2}}\right).$$

15. (Moldova TST 2015) Let $c \in \left(0, \frac{\pi}{2}\right)$,

$$a = \left(\frac{1}{sin(c)}\right)^{\frac{1}{cos^2(c)}}, b = \left(\frac{1}{cos(c)}\right)^{\frac{1}{sin^2(c)}}$$

. Prove that at least one of a, b is bigger than $\sqrt[11]{2015}$.

16. (Moldova TST 2015) Let a, b, c be positive real numbers such that $abc = 1$. Prove the following inequality:

$$a^3 + b^3 + c^3 + \frac{ab}{a^2 + b^2} + \frac{bc}{b^2 + c^2} + \frac{ca}{c^2 + a^2} \geq \frac{9}{2}$$

17. (All-Russian MO 2014) Does there exist positive $a \in \mathbf{R}$, such that

$$|\cos x| + |\cos ax| > \sin x + \sin ax$$

for all $x \in \mathbf{R}$?

18. (Balkan 2014) Let x, y and z be positive real numbers such that $xy + yz + xz = 3xyz$. Prove that

$$x^2y + y^2z + z^2x \geq 2(x + y + z) - 3$$

and determine when equality holds.

19. (Baltic Way 2014) Positive real numbers a, b, c satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$. Prove the inequality

$$\frac{1}{\sqrt{a^3 + b}} + \frac{1}{\sqrt{b^3 + c}} + \frac{1}{\sqrt{c^3 + a}} \leq \frac{3}{\sqrt{2}}.$$

20. (Benelux 2014) Find the smallest possible value of the expression

$$\left\lfloor \frac{a+b+c}{d} \right\rfloor + \left\lfloor \frac{b+c+d}{a} \right\rfloor + \left\lfloor \frac{c+d+a}{b} \right\rfloor + \left\lfloor \frac{d+a+b}{c} \right\rfloor$$

in which a, b, c, and d vary over the set of positive integers.

(Here $\lfloor x \rfloor$ denotes the biggest integer which is smaller than or equal to x.)

21. (Britain 2014) Prove that for $n \geq 2$ the following inequality holds:

$$\frac{1}{n+1}\left(1 + \frac{1}{3} + \ldots + \frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2} + \ldots + \frac{1}{2n}\right).$$

22. (Bosnia Herzegovina TST 2014) Let a,b and c be distinct real numbers. a) Determine value of

$$\frac{1+ab}{a-b} \cdot \frac{1+bc}{b-c} + \frac{1+bc}{b-c} \cdot \frac{1+ca}{c-a} + \frac{1+ca}{c-a} \cdot \frac{1+ab}{a-b}$$

b) Determine value of

$$\frac{1-ab}{a-b} \cdot \frac{1-bc}{b-c} + \frac{1-bc}{b-c} \cdot \frac{1-ca}{c-a} + \frac{1-ca}{c-a} \cdot \frac{1-ab}{a-b}$$

c) Prove the following ineqaulity

$$\frac{1+a^2b^2}{(a-b)^2}+\frac{1+b^2c^2}{(b-c)^2}+\frac{1+c^2a^2}{(c-a)^2}\geq\frac{3}{2}$$

When does quality holds?

23. (Canada 2014) Let $a_1, a_2, \ldots, a_n$ be positive real numbers whose product is 1. Show that the sum

$$\frac{a_1}{1+a_1}+\frac{a_2}{(1+a_1)(1+a_2)}+\frac{a_3}{(1+a_1)(1+a_2)(1+a_3)}+\cdots+\frac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)}$$

is greater than or equal to $\frac{2^n-1}{2^n}$.

24. (CentroAmerican 2014) Let a, b, c and d be real numbers such that no two of them are equal,

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}=4$$

and $ac=bd$. Find the maximum possible value of

$$\frac{a}{c}+\frac{b}{d}+\frac{c}{a}+\frac{d}{b}.$$

25. (China Girls Math Olympiad 2014) Let $x_1, x_2, \ldots, x_n$ be real numbers, where $n\geq 2$ is a given integer, and let $\lfloor x_1\rfloor, \lfloor x_2\rfloor, \ldots, \lfloor x_n\rfloor$ be a permutation of $1, 2, \ldots, n$. Find the maximum and minimum of

$$\sum_{i=1}^{n-1}\lfloor x_{i+1}-x_i\rfloor$$

(here $\lfloor x\rfloor$ is the largest integer not greater than x).

26. (China Northern MO 2014) Define a positive number sequence sequence $\{a_n\}$ by

$$a_1=1, (n^2+1)a_{n-1}^2=(n-1)^2a_n^2.$$

Prove that

$$\frac{1}{a_1^2}+\frac{1}{a_2^2}+\cdots+\frac{1}{a_n^2}\leq 1+\sqrt{1-\frac{1}{a_n^2}}.$$

27. (China Northern MO 2014) Let x, y, z, w be real numbers such that $x+2y+3z+4w=1$. Find the minimum of

$$x^2+y^2+z^2+w^2+(x+y+z+w)^2$$

28. (China TST 2014) For any real numbers sequence $\{x_n\}$,suppose that $\{y_n\}$ is a sequence such that: $y_1=x_1, y_{n+1}=x_{n+1}-(\sum\limits_{i=1}^{n}x_i^2)^{\frac{1}{2}}$ $(n\geq 1)$. Find the smallest positive number λ such that for any real numbers sequence $\{x_n\}$ and all positive integers m ,we have

$$\frac{1}{m}\sum_{i=1}^{m}x_i^2\leq\sum_{i=1}^{m}\lambda^{m-i}y_i^2.$$

29. (China TST 2014) Let n be a given integer which is greater than 1. Find the greatest constant $\lambda(n)$ such that for any non-zero complex $z_1, z_2, \cdots, z_n$,we have

$$\sum_{k=1}^{n} |z_k|^2 \geq \lambda(n) \min_{1\leq k\leq n} \{|z_{k+1} - z_k|^2\},$$

where $z_{n+1} = z_1$.

30. (China Western MO 2014) Let x, y be positive real numbers .Find the minimum of

$$x + y + \frac{|x-1|}{y} + \frac{|y-1|}{x}$$

.

31. (District Olympiad 2014) Prove that for any real numbers a and b the following inequality holds:

$$\left(a^2+1\right)\left(b^2+1\right) + 50 \geq 2\left(2a+1\right)\left(3b+1\right)$$

32. (ELMO Shortlist 2014) Given positive reals a, b, c, p, q satisfying $abc = 1$ and $p \geq q$, prove that

$$p\left(a^2+b^2+c^2\right) + q\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq (p+q)(a+b+c).$$

33. (ELMO Shortlist 2014) Let a, b, c, d, e, f be positive real numbers. Given that $def + de + ef + fd = 4$, show that

$$((a+b)de + (b+c)ef + (c+a)fd)^2 \geq\ 12(abde + bcef + cafd).$$

34. (ELMO Shortlist 2014) Let a, b, c be positive reals such that $a + b + c = ab + bc + ca$. Prove that

$$(a+b)^{ab-bc}(b+c)^{bc-ca}(c+a)^{ca-ab} \geq a^{ca}b^{ab}c^{bc}.$$

35. (ELMO Shortlist 2014) Let a, b, c be positive reals with $a^{2014} + b^{2014} + c^{2014} + abc = 4$. Prove that

$$\frac{a^{2013}+b^{2013}-c}{c^{2013}} + \frac{b^{2013}+c^{2013}-a}{a^{2013}} + \frac{c^{2013}+a^{2013}-b}{b^{2013}} \geq a^{2012}+b^{2012}+c^{2012}.$$

36. (ELMO Shortlist 2014) Let a, b, c be positive reals. Prove that

$$\sqrt{\frac{a^2(bc+a^2)}{b^2+c^2}} + \sqrt{\frac{b^2(ca+b^2)}{c^2+a^2}} + \sqrt{\frac{c^2(ab+c^2)}{a^2+b^2}} \geq a+b+c.$$

37. (Korea 2014) Suppose x, y, z are positive numbers such that $x+y+z = 1$. Prove that

$$\frac{(1+xy+yz+zx)(1+3x^3+3y^3+3z^3)}{9(x+y)(y+z)(z+x)} \geq \left(\frac{x\sqrt{1+x}}{\sqrt[4]{3+9x^2}} + \frac{y\sqrt{1+y}}{\sqrt[4]{3+9y^2}} + \frac{z\sqrt{1+z}}{\sqrt[4]{3+9z^2}}\right)^2.$$

38. (France TST 2014) Let n be a positive integer and $x_1, x_2, \ldots, x_n$ be positive reals. Show that there are numbers $a_1, a_2, \ldots, a_n \in \{-1, 1\}$ such that the following holds:

$$a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2 \geq (a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2$$

39. (Harvard-MIT Mathematics Tournament 2014) Find the largest real number c such that

$$\sum_{i=1}^{101} x_i^2 \geq cM^2$$

whenever $x_1, \ldots, x_{101}$ are real numbers such that $x_1 + \cdots + x_{101} = 0$ and M is the median of $x_1, \ldots, x_{101}$.

40. (India Regional MO 2014) Let a, b, c be positive real numbers such that

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \leq 1.$$

Prove that $(1+a^2)(1+b^2)(1+c^2) \geq 125$. When does equality hold?

41. (India Regional MO 2014) Let $x_1, x_2, x_3 \ldots x_{2014}$ be positive real numbers such that $\sum_{j=1}^{2014} x_j = 1$. Determine with proof the smallest constant K such that

$$K \sum_{j=1}^{2014} \frac{x_j^2}{1 - x_j} \geq 1$$

42. (IMO Training Camp 2014) Let a, b be positive real numbers.Prove that

$$(1+a)^8 + (1+b)^8 \geq 128ab(a+b)^2$$

43. (Iran 2014) Let x, y, z be three non-negative real numbers such that

$$x^2 + y^2 + z^2 = 2(xy + yz + zx).$$

Prove that

$$\frac{x+y+z}{3} \geq \sqrt[3]{2xyz}.$$

44. (Iran 2014) For any $a, b, c > 0$ satisfying $a + b + c + ab + ac + bc = 3$, prove that

$$2 \leq a + b + c + abc \leq 3$$

45. (Iran TST 2014) n is a natural number. for every positive real numbers $x_1, x_2, ..., x_{n+1}$ such that $x_1x_2...x_{n+1} = 1$ prove that:

$$\sqrt[x_1]{n} + ... + \sqrt[x_{n+1}]{n} \geq n^{\sqrt[n]{x_1}} + ... + n^{\sqrt[n]{x_{n+1}}}$$

46. (Iran TST 2014) if $x, y, z > 0$ are postive real numbers such that $x^2 + y^2 + z^2 = x^2y^2 + y^2z^2 + z^2x^2$ prove that

$$((x-y)(y-z)(z-x))^2 \leq 2((x^2-y^2)^2 + (y^2-z^2)^2 + (z^2-x^2)^2)$$

47. (Japan 2014) Suppose there exist $2m$ integers $i_1, i_2, \dots, i_m$ and $j_1, j_2, \dots, j_m$, of values in $\{1, 2, \dots, 1000\}$. These integers are not necessarily distinct. For any non-negative real numbers $a_1, a_2, \dots, a_{1000}$ satisfying $a_1+a_2+\cdots+a_{1000} = 1$, find the maximum positive integer m for which the following inequality holds

$$a_{i_1}a_{j_1} + a_{i_2}a_{j_2} + \cdots + a_{i_m}a_{j_m} \le \frac{1}{2.014}.$$

48. (Japan MO Finals 2014) Find the maximum value of real number k such that

$$\frac{a}{1+9bc+k(b-c)^2} + \frac{b}{1+9ca+k(c-a)^2} + \frac{c}{1+9ab+k(a-b)^2} \ge \frac{1}{2}$$

holds for all non-negative real numbers $a,\ b,\ c$ satisfying $a+b+c=1$.

49. (Turkey JBMO TST 2014) Determine the smallest value of

$$(a+5)^2 + (b-2)^2 + (c-9)^2$$

for all real numbers a, b, c satisfying $a^2+b^2+c^2-ab-bc-ca=3$

50. (JBMO 2014) For positive real numbers a, b, c with $abc = 1$ prove that

$$\left(a+\frac{1}{b}\right)^2 + \left(b+\frac{1}{c}\right)^2 + \left(c+\frac{1}{a}\right)^2 \ge 3(a+b+c+1)$$

51. (Korea 2014) Let x, y, z be the real numbers that satisfies the following.

$$(x-y)^2 + (y-z)^2 + (z-x)^2 = 8, x^3+y^3+z^3 = 1$$

Find the minimum value of

$$x^4 + y^4 + z^4$$

52. (Macedonia 2014) Let a, b, c be real numbers such that $a+b+c=4$ and $a, b, c > 1$. Prove that:

$$\frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{c-1} \ge \frac{8}{a+b} + \frac{8}{b+c} + \frac{8}{c+a}$$

53. (Mediterranean MO 2014) Let $a_1, \dots, a_n$ and $b_1 \dots, b_n$ be $2n$ real numbers. Prove that there exists an integer k with $1 \le k \le n$ such that

$$\sum_{i=1}^{n} |a_i - a_k| \quad \le \quad \sum_{i=1}^{n} |b_i - a_k|.$$

54. (Mexico 2014) Let a, b, c be positive reals such that $a+b+c=3$. Prove:

$$\frac{a^2}{a+\sqrt[3]{bc}} + \frac{b^2}{b+\sqrt[3]{ca}} + \frac{c^2}{c+\sqrt[3]{ab}} \ge \frac{3}{2}$$

And determine when equality holds.

55. (Middle European MO 2014) Determine the lowest possible value of the expression

$$\frac{1}{a+x}+\frac{1}{a+y}+\frac{1}{b+x}+\frac{1}{b+y}$$

where a, b, x, and y are positive real numbers satisfying the inequalities

$$\frac{1}{a+x} \geq \frac{1}{2}$$

$$\frac{1}{a+y} \geq \frac{1}{2}$$

$$\frac{1}{b+x} \geq \frac{1}{2}$$

$$\frac{1}{b+y} \geq 1.$$

56. (Moldova TST 2014) Let $a, b \in \mathbf{R}_+$ such that $a+b=1$. Find the minimum value of the following expression:

$$E(a,b)=3\sqrt{1+2a^2}+2\sqrt{40+9b^2}.$$

57. (Moldova TST 2014) Consider $n \geq 2$ positive numbers $0 < x_1 \leq x_2 \leq \ldots \leq x_n$, such that $x_1+x_2+\ldots+x_n=1$. Prove that if $x_n \leq \frac{2}{3}$, then there exists a positive integer $1 \leq k \leq n$ such that

$$\frac{1}{3} \leq x_1+x_2+\ldots+x_k < \frac{2}{3}$$

58. (Moldova TST 2014) Let a, b, c be positive real numbers such that $abc=1$. Determine the minimum value of

$$E(a,b,c)=\sum \frac{a^3+5}{a^3(b+c)}$$

59. (Romania TST 2014) Let a be a real number in the open interval $(0,1)$. Let $n \geq 2$ be a positive integer and let $f_n \colon \mathbf{R} \to \mathbf{R}$ be defined by $f_n(x) = x+\frac{x^2}{n}$. Show that

$$\frac{a(1-a)n^2+2a^2n+a^3}{(1-a)^2n^2+a(2-a)n+a^2} < (f_n \circ \cdots \circ f_n)(a) < \frac{an+a^2}{(1-a)n+a}$$

where there are n functions in the composition.

60. (Romania TST 2014) Determine the smallest real constant c such that

$$\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k}x_j\right)^2 \leq c\sum_{k=1}^{n}x_k^2$$

for all positive integers n and all positive real numbers $x_1, \cdots, x_n$.

61. (Romania TST 2014) Let n a positive integer and let $f\colon [0,1] \to \mathbf{R}$ an increasing function. Find the value of :

$$\max_{0\le x_1\le\cdots\le x_n\le 1} \sum_{k=1}^{n} f\left(\left|x_k - \frac{2k-1}{2n}\right|\right)$$

62. (Southeast MO 2014) Let $x_1, x_2, \cdots, x_n$ be non-negative real numbers such that $x_i x_j \le 4^{-|i-j|}$ $(1 \le i, j \le n)$. Prove that

$$x_1 + x_2 + \cdots + x_n \le \frac{5}{3}.$$

63. (Southeast MO 2014) Let $x_1, x_2, \cdots, x_n$ be positive real numbers such that $x_1 + x_2 + \cdots + x_n = 1$ $(n \ge 2)$. Prove that

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1} - x_{i+1}^3} \ge \frac{n^3}{n^2-1}.$$

here $x_{n+1} = x_1$.

64. (Turkey JBMO TST 2014) Prove that for positive reals a,b,c such that $a + b + c + abc = 4$,

$$\left(1 + \frac{a}{b} + ca\right)\left(1 + \frac{b}{c} + ab\right)\left(1 + \frac{c}{a} + bc\right) \ge 27$$

holds.

65. (Turkey TST 2014) Prove that for all all non-negative real numbers a, b, c with $a^2 + b^2 + c^2 = 1$

$$\sqrt{a+b} + \sqrt{a+c} + \sqrt{b+c} \ge 5abc + 2.$$

66. (Tuymaada MO 2014 Positive numbers a, b, c satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$. Prove the inequality

$$\frac{1}{\sqrt{a^3+1}} + \frac{1}{\sqrt{b^3+1}} + \frac{1}{\sqrt{c^3+1}} \le \frac{3}{\sqrt{2}}.$$

67. (USAJMO 2014) Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\min\left(\frac{10a^2-5a+1}{b^2-5b+10}, \frac{10b^2-5b+1}{c^2-5c+10}, \frac{10c^2-5c+1}{a^2-5a+10}\right) \le abc.$$

68. (USAMO 2014) Let a, b, c, d be real numbers such that $b - d \ge 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product

$$(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1)$$

can take.

69. (Uzbekistan 2014) For all $x, y, z \in \mathbf{R}\backslash\{1\}$, such that $xyz = 1$, prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

70. (Vietnam 2014) Find the maximum of

$$P = \frac{x^3y^4z^3}{(x^4+y^4)(xy+z^2)^3} + \frac{y^3z^4x^3}{(y^4+z^4)(yz+x^2)^3} + \frac{z^3x^4y^3}{(z^4+x^4)(zx+y^2)^3}$$

where x, y, z are positive real numbers.

71. (Albania TST 2013) Let a, b, c, d be positive real numbers such that $abcd = 1$.Find with proof that $x = 3$ is the minimal value for which the following inequality holds :

$$a^x + b^x + c^x + d^x \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$$

72. (All-Russian MO 2014) Let a, b, c, d be positive real numbers such that $2(a+b+c+d) \geq abcd$. Prove that

$$a^2 + b^2 + c^2 + d^2 \geq abcd.$$

73. (Baltic Way 2013) Prove that the following inequality holds for all positive real numbers x, y, z:

$$\frac{x^3}{y^2+z^2} + \frac{y^3}{z^2+x^2} + \frac{z^3}{x^2+y^2} \geq \frac{x+y+z}{2}.$$

74. (Bosnia Herzegovina TST 2013) Let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers of sum equal to 1. Let

$$F_n = x_1^2 + x_2^2 + \cdots + x_n^2 - 2(x_1x_2 + x_2x_3 + \cdots + x_nx_1)$$

. Find:

a) $\min F_3$;

b) $\min F_4$;

c) $\min F_5$.

75. (Canada 2013) Let x, y, z be real numbers that are greater than or equal to 0 and less than or equal to $\frac{1}{2}$

(a) Determine the minimum possible value of

$$x + y + z - xy - yz - zx$$

and determine all triples (x, y, z) for which this minimum is obtained. (b) Determine the maximum possible value of

$$x + y + z - xy - yz - zx$$

and determine all triples (x, y, z) for which this maximum is obtained.

76. (China Girls MO 2013) For any given positive numbers $a_1, a_2, \ldots, a_n$, prove that there exist positive numbers $x_1, x_2, \ldots, x_n$ satisfying

$$\sum_{i=1}^{n} x_i = 1$$

such that for any positive numbers $y_1, y_2, \ldots, y_n$ with

$$\sum_{i=1}^{n} y_i = 1$$

the inequality

$$\sum_{i=1}^{n} \frac{a_i x_i}{x_i + y_i} \geq \frac{1}{2} \sum_{i=1}^{n} a_i$$

holds.

77. (China 2013) Find all positive real numbers t with the following property: there exists an infinite set X of real numbers such that the inequality

$$\max\{|x - (a - d)|, |y - a|, |z - (a + d)|\} > td$$

holds for all (not necessarily distinct) $x, y, z \in X$, all real numbers a and all positive real numbers d.

78. (China Northern MO 2013) If $a_1, a_2, \cdots, a_{2013} \in [-2, 2]$ and

$$a_1 + a_2 + \cdots + a_{2013} = 0$$

, find the maximum of

$$a_1^3 + a_2^3 + \cdots + a_{2013}^3$$

.

79. (China TST 2013) Let n and k be two integers which are greater than 1. Let $a_1, a_2, \ldots, a_n, c_1, c_2, \ldots, c_m$ be non-negative real numbers such that i) $a_1 \geq a_2 \geq \ldots \geq a_n$ and $a_1 + a_2 + \ldots + a_n = 1$; ii) For any integer $m \in \{1, 2, \ldots, n\}$, we have that $c_1 + c_2 + \ldots + c_m \leq m^k$. Find the maximum of $c_1 a_1^k + c_2 a_2^k + \ldots + c_n a_n^k$.

80. (China TST 2013) Let $n > 1$ be an integer and let $a_0, a_1, \ldots, a_n$ be non-negative real numbers. Definite $S_k = \sum_{i=0}^{k} \binom{k}{i} a_i$ for $k = 0, 1, \ldots, n$. Prove that

$$\frac{1}{n} \sum_{k=0}^{n-1} S_k^2 - \frac{1}{n^2} \left(\sum_{k=0}^{n} S_k \right)^2 \leq \frac{4}{45} (S_n - S_0)^2.$$

81. (China TST 2013) Let $k \geq 2$ be an integer and let $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ be non-negative real numbers. Prove that

$$\left(\frac{n}{n-1} \right)^{n-1} \left(\frac{1}{n} \sum_{i=1}^{n} a_i^2 \right) + \left(\frac{1}{n} \sum_{i=1}^{n} b_i \right)^2 \geq \prod_{i=1}^{n} (a_i^2 + b_i^2)^{\frac{1}{n}}.$$

82. (China Western MO 2013) Let the integer $n \geq 2$, and the real numbers $x_1, x_2, \cdots, x_n \in [0,1]$.Prove that

$$\sum_{1 \leq k < j \leq n} k x_k x_j \leq \frac{n-1}{3} \sum_{k=1}^{n} k x_k.$$

83. (District Olympiad 2013) Let $n \in \mathbf{N}^*$ and $a_1, a_2, ..., a_n \in \mathbf{R}$ so $a_1 + a_2 + ... + a_k \leq k, (\forall)\, k \in \{1, 2, ..., n\}$ .Prove that

$$\frac{a_1}{1} + \frac{a_2}{2} + ... + \frac{a_n}{n} \leq \frac{1}{1} + \frac{1}{2} + ... + \frac{1}{n}$$

84. (District Olympiad 2013) Let $a, b \in \mathbf{C}$. Prove that $|az + b\bar{z}| \leq 1$, for every $z \in \mathbf{C}$, with $|z| = 1$, if and only if $|a| + |b| \leq 1$.

85. (ELMO 2013) Let $a_1, a_2, ..., a_9$ be nine real numbers, not necessarily distinct, with average m. Let A denote the number of triples $1 \leq i < j < k \leq 9$ for which

$$a_i + a_j + a_k \geq 3m$$

. What is the minimum possible value of A?

86. (ELMO 2013) Let a, b, c be positive reals satisfying $a+b+c = \sqrt[7]{a} + \sqrt[7]{b} + \sqrt[7]{c}$. Prove that

$$a^a b^b c^c \geq 1$$

87. (ELMO Shortlist 2013) Prove that for all positive reals a, b, c,

$$\frac{1}{a + \frac{1}{b} + 1} + \frac{1}{b + \frac{1}{c} + 1} + \frac{1}{c + \frac{1}{a} + 1} \geq \frac{3}{\sqrt[3]{abc} + \frac{1}{\sqrt[3]{abc}} + 1}.$$

88. (ELMO Shortlist 2013) Positive reals a, b, and c obey $\frac{a^2+b^2+c^2}{ab+bc+ca} = \frac{ab+bc+ca+1}{2}$. Prove that

$$\sqrt{a^2 + b^2 + c^2} \leq 1 + \frac{|a-b| + |b-c| + |c-a|}{2}.$$

89. (ELMO Shortlist 2013) Let a, b, c be positive reals such that $a+b+c = 3$. Prove that

$$18 \sum_{\text{cyc}} \frac{1}{(3-c)(4-c)} + 2(ab + bc + ca) \geq 15.$$

90. (ELMO Shortlist 2013) Let a, b, c be positive reals with $a^{2014} + b^{2014} + c^{2014} + abc = 4$. Prove that

$$\frac{a^{2013} + b^{2013} - c}{c^{2013}} + \frac{b^{2013} + c^{2013} - a}{a^{2013}} + \frac{c^{2013} + a^{2013} - b}{b^{2013}} \geq a^{2012} + b^{2012} + c^{2012}.$$

91. (ELMO Shortlist 2013) Let a, b, c be positive reals, and let

$$\sqrt[2013]{\frac{3}{a^{2013}+b^{2013}+c^{2013}}} = P$$

Prove that

$$\prod_{\text{cyc}} \left(\frac{(2P + \frac{1}{2a+b})(2P + \frac{1}{a+2b})}{(2P + \frac{1}{a+b+c})^2} \right) \geq \prod_{\text{cyc}} \left(\frac{(P + \frac{1}{4a+b+c})(P + \frac{1}{3b+3c})}{(P + \frac{1}{3a+2b+c})(P + \frac{1}{3a+b+2c})} \right).$$

92. (Federal Competition for Advanced students 2013) For a positive integer n, let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that for all real numbers $x_1 > x_2 > \ldots > x_n > 0$ with $x_1 + x_2 + \ldots + x_n < 1$, the inequality

$$\sum_{k=1}^{n} a_k x_k^3 < 1$$

holds. Show that

$$na_1 + (n-1)a_2 + \ldots + (n-j+1)a_j + \ldots + a_n \leq \frac{n^2(n+1)^2}{4}.$$

93. (Korea 2013) For a positive integer $n \geq 2$, define set $T = \{(i, j) | 1 \leq i < j \leq n, i|j\}$. For nonnegative real numbers $x_1, x_2, \cdots, x_n$ with $x_1 + x_2 + \cdots + x_n = 1$, find the maximum value of

$$\sum_{(i,j)\in T} x_i x_j$$

in terms of n.

94. (Hong Kong 2013) Let a, b, c be positive real numbers such that $ab+bc+ca = 1$. Prove that

$$\sqrt[4]{\frac{\sqrt{3}}{a} + 6\sqrt{3}b} + \sqrt[4]{\frac{\sqrt{3}}{b} + 6\sqrt{3}c} + \sqrt[4]{\frac{\sqrt{3}}{c} + 6\sqrt{3}a} \leq \frac{1}{abc}$$

When does inequality hold?

95. (IMC 2013) Let z be a complex number with $|z + 1| > 2$. Prove that

$$\left|z^3 + 1\right| > 1$$

.

96. (India Regional MO 2013) Given real numbers $a, b, c, d, e > 1$. Prove that

$$\frac{a^2}{c-1} + \frac{b^2}{d-1} + \frac{c^2}{e-1} + \frac{d^2}{a-1} + \frac{e^2}{b-1} \geq 20$$

97. (Iran TST 2013) Let a, b, c be sides of a triangle such that $a \geq b \geq c$. prove that:

$$\sqrt{a(a+b-\sqrt{ab})} + \sqrt{b(a+c-\sqrt{ac})} + \sqrt{c(b+c-\sqrt{bc})} \geq a+b+c$$

98. (Macedonia JBMO TST 2013) $a, b, c > 0$ and $abc = 1$. Prove that

$$\frac{1}{2}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) + \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \geq 3$$

.

99. (Turkey JBMO TST 2013) For all positive real numbers a, b, c satisfying $a + b + c = 1$, prove that

$$\frac{a^4 + 5b^4}{a(a+2b)} + \frac{b^4 + 5c^4}{b(b+2c)} + \frac{c^4 + 5a^4}{c(c+2a)} \geq 1 - ab - bc - ca$$

100. (Turkey JBMO TST 2013) Let a, b, c, d be real numbers greater than 1 and x, y be real numbers such that

$$a^x + b^y = (a^2 + b^2)^x \quad \text{and} \quad c^x + d^y = 2^y (cd)^{y/2}$$

Prove that $x < y$.

101. (JBMO 2013) Show that

$$\left(a + 2b + \frac{2}{a+1}\right)\left(b + 2a + \frac{2}{b+1}\right) \geq 16$$

for all positive real numbers a and b such that $ab \geq 1$.

102. (Kazakhstan 2013) Find maximum value of

$$|a^2 - bc + 1| + |b^2 - ac + 1| + |c^2 - ba + 1|$$

when a, b, c are reals in $[-2; 2]$.

103. (Kazakhstan 2013) Consider the following sequence $a_1 = 1; a_n = \frac{a_{[\frac{n}{2}]}}{2} + \frac{a_{[\frac{n}{3}]}}{3} + \ldots + \frac{a_{[\frac{n}{n}]}}{n}$ Prove that $\forall n \in \mathbf{N}$

$$a_{2n} < 2a_n$$

104. (Korea 2013) Let $a, b, c > 0$ such that $ab + bc + ca = 3$. Prove that

$$\sum_{cyc} \frac{(a+b)^3}{(2(a+b)(a^2+b^2))^{\frac{1}{3}}} \geq 12$$

105. (Kosovo 2013) For all real numbers a prove that

$$3(a^4 + a^2 + 1) \geq (a^2 + a + 1)^2$$

106. (Kosovo 2013) Which number is bigger $\sqrt[2012]{2012!}$ or $\sqrt[2013]{2013!}$?

107. (Macedonia 2013) Let a, b, c be positive real numbers such that $a^4 + b^4 + c^4 = 3$. Prove that

$$\frac{9}{a^2 + b^4 + c^6} + \frac{9}{a^4 + b^6 + c^2} + \frac{9}{a^6 + b^2 + c^4} \leq a^6 + b^6 + c^6 + 6$$

108. (Mediterranean MO 2013) Let x, y, z be positive reals for which:

$$\sum (xy)^2 = 6xyz$$

Prove that:

$$\sum \sqrt{\frac{x}{x+yz}} \geq \sqrt{3}$$

.

109. (Middle European MO 2013) Let a, b, c be positive real numbers such that

$$a+b+c = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Prove that

$$2(a+b+c) \geq \sqrt[3]{7a^2b+1} + \sqrt[3]{7b^2c+1} + \sqrt[3]{7c^2a+1}.$$

Find all triples (a, b, c) for which equality holds.

110. (Middle European MO 2013) Let x, y, z, w be nonzero real numbers such that $x+y \neq 0$, $z+w \neq 0$, and $xy+zw \geq 0$. Prove that

$$\left(\frac{x+y}{z+w} + \frac{z+w}{x+y}\right)^{-1} + \frac{1}{2} \geq \left(\frac{x}{z} + \frac{z}{x}\right)^{-1} + \left(\frac{y}{w} + \frac{w}{y}\right)^{-1}$$

111. (Moldova TST 2013) For any positive real numbers x, y, z, prove that

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq \frac{z(x+y)}{y(y+z)} + \frac{x(z+y)}{z(x+z)} + \frac{y(x+z)}{x(x+y)}$$

112. (Moldova TST 2013) Prove that for any positive real numbers a_i, b_i, c_i with $i = 1, 2, 3$,

$$(a_1^3+b_1^3+c_1^3+1)(a_2^3+b_2^3+c_2^3+1)(a_3^3+b_3^3+c_3^3+1) \geq \frac{3}{4}(a_1+b_1+c_1)(a_2+b_2+c_2)(a_3+b_3+c_3)$$

113. (Moldova TST 2013) Consider real numbers x, y, z such that $x, y, z > 0$. Prove that

$$(xy+yz+xz)\left(\frac{1}{x^2+y^2} + \frac{1}{x^2+z^2} + \frac{1}{y^2+z^2}\right) > \frac{5}{2}.$$

114. (Olympic Revenge 2013) Let a, b, c, d to be non negative real numbers satisfying $ab+ac+ad+bc+bd+cd = 6$. Prove that

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} + \frac{1}{d^2+1} \geq 2$$

115. (Philippines 2013) Let r and s be positive real numbers such that

$$(r+s-rs)(r+s+rs) = rs$$

. Find the minimum value of $r+s-rs$ and $r+s+rs$

116. (Poland 2013) Let k,m and n be three different positive integers. Prove that

$$\left(k-\frac{1}{k}\right)\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right) \le kmn-(k+m+n).$$

117. (Rioplatense 2013) Let a, b, c, d be real positive numbers such that

$$a^2+b^2+c^2+d^2=1$$

Prove that

$$(1-a)(1-b)(1-c)(1-d) \ge abcd$$

118. (Romania 2013) To be considered the following complex and distinct a, b, c, d. Prove that the following affirmations are equivalent:

i) For every $z \in \mathbf{C}$ this inequality takes place :

$$|z-a|+|z-b| \ge |z-c|+|z-d|$$

ii) There is $t \in (0,1)$ so that $c = ta + (1-t)\,b$ si $d = (1-t)\,a + tb$

119. (Romania 2013)

a)Prove that

$$\frac{1}{2}+\frac{1}{3}+...+\frac{1}{2^m} < m$$

for any $m \in \mathbf{N}^*$.

b)Let $p_1, p_2, ..., p_n$ be the prime numbers less than 2^{100}. Prove that

$$\frac{1}{p_1}+\frac{1}{p_2}+...+\frac{1}{p_n} < 10$$

120. (Romania TST 2013) Let n be a positive integer and let $x_1, \ldots, x_n$ be positive real numbers. Show that:

$$\min\left(x_1, \frac{1}{x_1}+x_2, \cdots, \frac{1}{x_{n-1}}+x_n, \frac{1}{x_n}\right) \le 2\cos\frac{\pi}{n+2} \le \max\left(x_1, \frac{1}{x_1}+x_2, \cdots, \frac{1}{x_{n-1}}+x_n, \frac{1}{x_n}\right).$$

121. (Serbia 2013) Find the largest constant $K \in \mathbf{R}$ with the following property: if $a_1, a_2, a_3, a_4 > 0$ are numbers satisfying

$$a_i^2+a_j^2+a_k^2 \ge 2(a_ia_j+a_ja_k+a_ka_i)$$

for every $1 \le i < j < k \le 4$, then

$$a_1^2+a_2^2+a_3^2+a_4^2 \ge K(a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4).$$

122. (Southeast MO 2013) Let a, b be real numbers such that the equation $x^3-ax^2+bx-a=0$ has three positive real roots . Find the minimum of

$$\frac{2a^3-3ab+3a}{b+1}$$

123. (Southeast MO 2013) $n \geq 3$ is a integer. $\alpha, \beta, \gamma \in (0,1)$. For every $a_k, b_k, c_k \geq 0 (k = 1, 2, \ldots, n)$ with

$$\sum_{k=1}^{n}(k+\alpha)a_k \leq \alpha,\ \sum_{k=1}^{n}(k+\beta)b_k \leq \beta,\ \sum_{k=1}^{n}(k+\gamma)c_k \leq \gamma$$

we always have

$$\sum_{k=1}^{n}(k+\lambda)a_k b_k c_k \leq \lambda$$

Find the minimum of λ

124. (Today's Calculation of Integrals 2013) Let $m,\ n$ be positive integer such that $2 \leq m < n$.

(1) Prove the inequality as follows.

$$\frac{n+1-m}{m(n+1)} < \frac{1}{m^2} + \frac{1}{(m+1)^2} + \cdots + \frac{1}{(n-1)^2} + \frac{1}{n^2} < \frac{n+1-m}{n(m-1)}$$

(2) Prove the inequality as follows.

$$\frac{3}{2} \leq \lim_{n\to\infty}\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right) \leq 2$$

(3) Prove the inequality which is made precisely in comparison with the inequality in (2) as follows.

$$\frac{29}{18} \leq \lim_{n\to\infty}\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right) \leq \frac{61}{36}$$

125. (Tokyo University Entrance Exam 2013) Let $a,\ b$ be real constants. If real numbers $x,\ y$ satisfy $x^2 + y^2 \leq 25,\quad 2x + y \leq 5$, then find the minimum value of

$$z = x^2 + y^2 - 2ax - 2by$$

126. (Turkey Junior MO 2013) Let x, y, z be real numbers satisfying $x+y+z = 0$ and $x^2 + y^2 + z^2 = 6$. Find the maximum value of

$$|(x-y)(y-z)(z-x)|$$

127. (Turkey 2013) Find the maximum value of M for which for all positive real numbers a, b, c we have

$$a^3 + b^3 + c^3 - 3abc \geq M(ab^2 + bc^2 + ca^2 - 3abc)$$

128. (Turkey TST 2013) For all real numbers x, y, z such that $-2 \leq x, y, z \leq 2$ and $x^2 + y^2 + z^2 + xyz = 4$, determine the least real number K satisfying

$$\frac{z(xz + yz + y)}{xy + y^2 + z^2 + 1} \leq K.$$

129. (Tuymaada 2013) Prove that if x, y, z are positive real numbers and $xyz = 1$ then

$$\frac{x^3}{x^2+y} + \frac{y^3}{y^2+z} + \frac{z^3}{z^2+x} \geq \frac{3}{2}.$$

130. (Tuymaada 2013) For every positive real numbers a and b prove the inequality

$$\sqrt{ab} \leq \frac{1}{3}\sqrt{\frac{a^2+b^2}{2}} + \frac{2}{3}\frac{2}{\frac{1}{a}+\frac{1}{b}}.$$

131. (USAMTS 2013) An infinite sequence of real numbers $a_1, a_2, a_3, \dots$ is called *spooky* if $a_1 = 1$ and for all integers $n > 1$,

$$\begin{array}{l} na_1 + (n-1)a_2 + (n-2)a_3 + \dots + 2a_{n-1} + a_n < 0, \\ n^2a_1 + (n-1)^2a_2 + (n-2)^2a_3 + \dots + 2^2a_{n-1} + a_n > 0. \end{array}$$

Given any *spooky* sequence $a_1, a_2, a_3, \dots$, prove that

$$2013^3a_1 + 2012^3a_2 + 2011^3a_3 + \dots + 2^3a_{2012} + a_{2013} < 12345.$$

132. (Uzbekistan 2013) Let real numbers a, b such that $a \geq b \geq 0$. Prove that

$$\sqrt{a^2+b^2} + \sqrt[3]{a^3+b^3} + \sqrt[4]{a^4+b^4} \leq 3a + b.$$

133. (Uzbekistan 2013) Let x and y are real numbers such that $x^2y^2 + 2yx^2 + 1 = 0$. If

$$S = \frac{2}{x^2} + 1 + \frac{1}{x} + y(y + 2 + \frac{1}{x})$$

find

(a)max$S$

(b) min$S$.

134. (Albania TST 2012) Find the greatest value of the expression

$$\frac{1}{x^2-4x+9} + \frac{1}{y^2-4y+9} + \frac{1}{z^2-4z+9}$$

where x, y, z are nonnegative real numbers such that $x + y + z = 1$.

135. (All-Russian MO 2012) The positive real numbers $a_1, \dots, a_n$ and k are such that

$$a_1 + \dots + a_n = 3k$$
$$a_1^2 + \dots + a_n^2 = 3k^2$$

and

$$a_1^3 + \dots + a_n^3 > 3k^3 + k$$

Prove that the difference between some two of $a_1, \dots, a_n$ is greater than 1.

136. (All-Russian MO 2012) Any two of the real numbers a_1, a_2, a_3, a_4, a_5 differ by no less than 1. There exists some real number k satisfying

$$a_1 + a_2 + a_3 + a_4 + a_5 = 2k$$

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = 2k^2$$

Prove that $k^2 \geq \frac{25}{3}$.

137. (APMO 2012) Let n be an integer greater than or equal to 2. Prove that if the real numbers $a_1, a_2, \cdots, a_n$ satisfy $a_1^2 + a_2^2 + \cdots + a_n^2 = n$, then

$$\sum_{1 \leq i < j \leq n} \frac{1}{n - a_i a_j} \leq \frac{n}{2}$$

must hold.

138. (Balkan 2012) Prove that

$$\sum_{cyc} (x+y)\sqrt{(z+x)(z+y)} \geq 4(xy + yz + zx),$$

for all positive real numbers x, y and z.

139. (Baltic Way 2012) Let a, b, c be real numbers. Prove that

$$ab + bc + ca + \max\{|a-b|, |b-c|, |c-a|\} \leq 1 + \frac{1}{3}(a+b+c)^2.$$

140. (Bosnia Herzegovina TST 2012) Prove for all positive real numbers a, b, c, such that $a^2 + b^2 + c^2 = 1$:

$$\frac{a^3}{b^2 + c} + \frac{b^3}{c^2 + a} + \frac{c^3}{a^2 + b} \geq \frac{\sqrt{3}}{1 + \sqrt{3}}.$$

141. (Canada 2012) Let x, y and z be positive real numbers. Show that

$$x^2 + xy^2 + xyz^2 \geq 4xyz - 4$$

142. (CentroAmerican 2012) Let a, b, c be real numbers that satisfy $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{a+c} = 1$ and $ab + bc + ac > 0$.
Show that

$$a + b + c - \frac{abc}{ab + bc + ac} \geq 4$$

143. (China Girls Math Olympiad 2012) Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers. Prove that

$$\frac{1}{1+a_1} + \frac{a_1}{(1+a_1)(1+a_2)} + \frac{a_1 a_2}{(1+a_1)(1+a_2)(1+a_3)} + \cdots + \frac{a_1 a_2 \cdots a_{n-1}}{(1+a_1)(1+a_2)\cdots(1+a_n)} \leq 1.$$

144. (China 2012) Let $f(x) = (x+a)(x+b)$ where $a, b > 0$. For any reals $x_1, x_2, \ldots, x_n \geq 0$ satisfying $x_1 + x_2 + \ldots + x_n = 1$, find the maximum of

$$F = \sum_{1 \leq i < j \leq n} \min\{f(x_i), f(x_j)\}$$

145. (China 2012) Suppose that $x, y, z \in [0,1]$. Find the maximal value of the expression
$$\sqrt{|x-y|}+\sqrt{|y-z|}+\sqrt{|z-x|}.$$

146. (china TST 2012) Complex numbers x_i, y_i satisfy $|x_i| = |y_i| = 1$ for $i = 1, 2, \ldots, n$. Let $x = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$, $y = \frac{1}{n}\sum\limits_{i=1}^{n} y_i$ and $z_i = xy_i + yx_i - x_iy_i$. Prove that
$$\sum_{i=1}^{n} |z_i| \leq n$$
.

147. (China TST 2012) Given two integers m, n which are greater than 1. r, s are two given positive real numbers such that $r < s$. For all $a_{ij} \geq 0$ which are not all zeroes,find the maximal value of the expression
$$f = \frac{(\sum_{j=1}^{n}(\sum_{i=1}^{m} a_{ij}^s)^{\frac{r}{s}})^{\frac{1}{r}}}{(\sum_{i=1}^{m})\sum_{j=1}^{n} a_{ij}^r)^{\frac{s}{r}})^{\frac{1}{s}}}.$$

148. (China TST 2012) Given an integer $k \geq 2$. Prove that there exist k pairwise distinct positive integers $a_1, a_2, \ldots, a_k$ such that for any non-negative integers $b_1, b_2, \ldots, b_k, c_1, c_2, \ldots, c_k$ satisfying $a_1 \leq b_i \leq 2a_i, i = 1, 2, \ldots, k$ and $\prod_{i=1}^{k} b_i^{c_i} < \prod_{i=1}^{k} b_i$, we have
$$k\prod_{i=1}^{k} b_i^{c_i} < \prod_{i=1}^{k} b_i.$$

149. (Czech-Polish-Slovak MAtch 2012) Let a, b, c, d be positive real numbers such that
$$abcd = 4, \qquad a^2 + b^2 + c^2 + d^2 = 10$$

Find the maximum possible value of
$$ab + bc + cd + da$$

150. (ELMO Shortlist 2012) Let $x_1, x_2, x_3, y_1, y_2, y_3$ be nonzero real numbers satisfying $x_1 + x_2 + x_3 = 0, y_1 + y_2 + y_3 = 0$. Prove that
$$\frac{x_1x_2 + y_1y_2}{\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}}+\frac{x_2x_3 + y_2y_3}{\sqrt{(x_2^2+y_2^2)(x_3^2+y_3^2)}}+\frac{x_3x_1 + y_3y_1}{\sqrt{(x_3^2+y_3^2)(x_1^2+y_1^2)}} \geq -\frac{3}{2}.$$

151. (ELMO Shortlist 2012) Let a, b, c be three positive real numbers such that $a \leq b \leq c$ and $a + b + c = 1$. Prove that
$$\frac{a+c}{\sqrt{a^2+c^2}}+\frac{b+c}{\sqrt{b^2+c^2}}+\frac{a+b}{\sqrt{a^2+b^2}} \leq \frac{3\sqrt{6}(b+c)^2}{\sqrt{(a^2+b^2)(b^2+c^2)(c^2+a^2)}}.$$

152. (ELMO Shortlist 2012) Let $a, b, c \geq 0$. Show that
$$(a^2+2bc)^{2012}+(b^2+2ca)^{2012}+(c^2+2ab)^{2012} \leq (a^2+b^2+c^2)^{2012}+2(ab+bc+ca)^{2012}$$

153. (ELMO Shortlist 2012) Let a, b, c be distinct positive real numbers, and let k be a positive integer greater than 3. Show that

$$\left|\frac{a^{k+1}(b-c)+b^{k+1}(c-a)+c^{k+1}(a-b)}{a^k(b-c)+b^k(c-a)+c^k(a-b)}\right| \geq \frac{k+1}{3(k-1)}(a+b+c)$$

and

$$\left|\frac{a^{k+2}(b-c)+b^{k+2}(c-a)+c^{k+2}(a-b)}{a^k(b-c)+b^k(c-a)+c^k(a-b)}\right| \geq \frac{(k+1)(k+2)}{3k(k-1)}(a^2+b^2+c^2).$$

154. (Federal competition for advanced students 2012) Determine the maximum value of m, such that the inequality

$$(a^2+4(b^2+c^2))(b^2+4(a^2+c^2))(c^2+4(a^2+b^2)) \geq m$$

holds for every $a, b, c \in \mathbf{R} \setminus \{0\}$ with $\left|\frac{1}{a}\right| + \left|\frac{1}{b}\right| + \left|\frac{1}{c}\right| \leq 3$. When does equality occur?

155. (Korea 2012) Let x, y, z be positive real numbers. Prove that

$$\frac{2x^2+xy}{(y+\sqrt{zx}+z)^2} + \frac{2y^2+yz}{(z+\sqrt{xy}+x)^2} + \frac{2z^2+zx}{(x+\sqrt{yz}+y)^2} \geq 1$$

156. (Finnish National High School Math Competition 2012) Let $k, n \in \mathbf{N}, 0 < k < n$. Prove that

$$\sum_{j=1}^{k} \binom{n}{j} = \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{k} \leq n^k.$$

157. (IMO 2012) Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n > n^n.$$

158. (India Regional MO 2012) Let a and b be positive real numbers such that $a+b=1$. Prove that

$$a^a b^b + a^b b^a \leq 1$$

159. (India Regional MO 2012) Let a, b, c be positive real numbers such that $abc(a+b+c)=3$. Prove that we have

$$(a+b)(b+c)(c+a) \geq 8.$$

Also determine the case of equality.

160. (Iran TST 2012) For positive reals a, b and c with $ab+bc+ca=1$, show that

$$\sqrt{3}(\sqrt{a}+\sqrt{b}+\sqrt{c}) \leq \frac{a\sqrt{a}}{bc} + \frac{b\sqrt{b}}{ca} + \frac{c\sqrt{c}}{ab}.$$

161. (JBMO 2012) Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{a}{b}+\frac{a}{c}+\frac{c}{b}+\frac{c}{a}+\frac{b}{c}+\frac{b}{a}+6 \geq 2\sqrt{2}\left(\sqrt{\frac{1-a}{a}}+\sqrt{\frac{1-b}{b}}+\sqrt{\frac{1-c}{c}}\right).$$

When does equality hold?

162. (JBMO shortlist 2012) Let a , b , c be positive real numbers such that $abc=1$. Show that :

$$\frac{1}{a^3+bc}+\frac{1}{b^3+ca}+\frac{1}{c^3+ab} \leq \frac{(ab+bc+ca)^2}{6}$$

163. (JBMO shortlist 2012) Let a , b , c be positive real numbers such that $a+b+c=a^2+b^2+c^2$. Prove that :

$$\frac{a^2}{a^2+ab}+\frac{b^2}{b^2+bc}+\frac{c^2}{c^2+ca} \geq \frac{a+b+c}{2}$$

164. (JBMO shortlist 2012) Find the largest positive integer n for which the inequality

$$\frac{a+b+c}{abc+1}+\sqrt[n]{abc} \leq \frac{5}{2}$$

holds true for all $a, b, c \in [0,1]$. Here we make the convention $\sqrt[1]{abc}=abc$.

165. (Macedonia JBMO TST 2012) Let a,b,c be positive real numbers and $a+b+c+2=abc$. Prove that

$$\frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1} \geq 2.$$

166. (Turkey JBMO TST 2012) Find the greatest real number M for which

$$a^2+b^2+c^2+3abc \geq M(ab+bc+ca)$$

for all non-negative real numbers a, b, c satisfying $a+b+c=4$.

167. (Turkey JBMO TST 2012) Show that for all real numbers x, y satisfying $x+y \geq 0$

$$(x^2+y^2)^3 \geq 32(x^3+y^3)(xy-x-y)$$

168. (Moldova JBMO TST 2012) Let $1 \leq a, b, c, d, e, f, g, h, k \leq 9$ and $a, b, c, d, e, f, g, h, k$ are different integers, find the minimum value of the expression

$$E = abc+def+ghk$$

and prove that it is minimum.

169. (Moldova JBMO TST 2012) Let a, b, c be positive real numbers, prove the inequality:

$$(a+b+c)^2+ab+bc+ac \geq 6\sqrt{abc(a+b+c)}$$

170. (Kazakhstan 2012) Let $a, b, c, d > 0$ for which the following conditions:

a) $(a-c)(b-d) = -4$

b) $\frac{a+c}{2} \geq \frac{a^2+b^2+c^2+d^2}{a+b+c+d}$

Find the minimum of expression $a+c$

171. (Kazakhstan 2012) For a positive reals $x_1, ..., x_n$ prove inequality:

$$\frac{1}{x_1+1} + ... + \frac{1}{x_n+1} \leq \frac{n}{1+\frac{n}{\frac{1}{x_1}+...+\frac{1}{x_n}}}$$

172. (Korea 2012) a, b, c are positive numbers such that $a^2+b^2+c^2 = 2abc+1$. Find the maximum value of

$$(a-2bc)(b-2ca)(c-2ab)$$

173. (Korea 2012) Let $\{a_1, a_2, \cdots, a_{10}\} = \{1, 2, \cdots, 10\}$. Find the maximum value of

$$\sum_{n=1}^{10}(na_n^2 - n^2a_n)$$

174. (Kyoto University Entry Examination 2012) When real numbers x, y moves in the constraint with $x^2+xy+y^2 = 6$. Find the range of

$$x^2y + xy^2 - x^2 - 2xy - y^2 + x + y.$$

175. (Kyrgyzstan 2012) Given positive real numbers $a_1, a_2, ..., a_n$ with $a_1+a_2+ ... + a_n = 1$. Prove that

$$\left(\frac{1}{a_1^2}-1\right)\left(\frac{1}{a_2^2}-1\right)...\left(\frac{1}{a_n^2}-1\right) \geq (n^2-1)^n$$

176. (Macedonia 2012) If a, b, c, d are positive real numbers such that $abcd = 1$ then prove that the following inequality holds

$$\frac{1}{bc+cd+da-1}+\frac{1}{ab+cd+da-1}+\frac{1}{ab+bc+da-1}+\frac{1}{ab+bc+cd-1} \leq 2\,.$$

When does inequality hold?

177. (Middle European MO 2012) Let a, b and c be positive real numbers with $abc = 1$. Prove that

$$\sqrt{9+16a^2} + \sqrt{9+16b^2} + \sqrt{9+16c^2} \geq 3 + 4(a+b+c)$$

178. (Olympic Revenge 2012) Let $x_1, x_2, \ldots, x_n$ positive real numbers. Prove that:

$$\sum_{cyc}\frac{1}{x_i^3 + x_{i-1}x_ix_{i+1}} \leq \sum_{cyc}\frac{1}{x_ix_{i+1}(x_i+x_{i+1})}$$

179. (Pre-Vietnam MO 2012) For $a, b, c > 0 : \ abc = 1$ prove that

$$a^3 + b^3 + c^3 + 6 \geq (a + b + c)^2$$

180. (Puerto rico TST 2012) Let x, y and z be consecutive integers such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} > \frac{1}{45}.$$

Find the maximum value of

$$x + y + z$$

181. (Regional competition for advanced students 2012) Prove that the inequality

$$a + a^3 - a^4 - a^6 < 1$$

holds for all real numbers a.

182. (Romania 2012) Prove that if $n \geq 2$ is a natural number and $x_1, x_2, \ldots, x_n$ are positive real numbers, then:

$$4\left(\frac{x_1^3 - x_2^3}{x_1 + x_2} + \frac{x_2^3 - x_3^3}{x_2 + x_3} + \ldots + \frac{x_{n-1}^3 - x_n^3}{x_{n-1} + x_n} + \frac{x_n^3 - x_1^3}{x_n + x_1}\right) \leq$$

$$\leq (x_1 - x_2)^2 + (x_2 - x_3)^2 + \ldots + (x_{n-1} - x_n)^2 + (x_n - x_1)^2$$

183. (Romania 2012) Let a , b and c be three complex numbers such that $a + b + c = 0$ and $|a| = |b| = |c| = 1$. Prove that:

$$3 \leq |z - a| + |z - b| + |z - c| \leq 4,$$

for any $z \in \mathbf{C}$, $|z| \leq 1$.

184. (Romania 2012) Let $a, b \in \mathbf{R}$ with $0 < a < b$. Prove that:

a)

$$2\sqrt{ab} \leq \frac{x + y + z}{3} + \frac{ab}{\sqrt[3]{xyz}} \leq a + b$$

for $x, y, z \in [a, b]$.

b)

$$\{\frac{x + y + z}{3} + \frac{ab}{\sqrt[3]{xyz}} \mid x, y, z \in [a, b]\} = [2\sqrt{ab}, a + b] \,.$$

185. (Romania TST 2012) Let k be a positive integer. Find the maximum value of

$$a^{3k-1}b + b^{3k-1}c + c^{3k-1}a + k^2 a^k b^k c^k,$$

where a, b, c are non-negative reals such that $a + b + c = 3k$.

186. (Romania TST 2012) Let $f, g : \mathbf{Z} \to [0, \infty)$ be two functions such that $f(n) = g(n) = 0$ with the exception of finitely many integers n. Define $h : \mathbf{Z} \to [0, \infty)$ by

$$h(n) = \max\{f(n-k)g(k) : k \in \mathbf{Z}\}.$$

Let p and q be two positive reals such that $1/p + 1/q = 1$. Prove that

$$\sum_{n \in \mathbf{Z}} h(n) \geq \left(\sum_{n \in \mathbf{Z}} f(n)^p\right)^{1/p} \left(\sum_{n \in \mathbf{Z}} g(n)^q\right)^{1/q}.$$

187. (South East MO 2012) Let a, b, c, d be real numbers satisfying inequality

$$a \cos x + b \cos 2x + c \cos 3x + d \cos 4x \leq 1$$

holds for any real number x. Find the maximal value of

$$a + b - c + d$$

and determine the values of a, b, c, d when that maximum is attained.

188. (South East MO 2012) Find the least natural number n, such that the following inequality holds:

$$\sqrt{\frac{n-2011}{2012}} - \sqrt{\frac{n-2012}{2011}} < \sqrt[3]{\frac{n-2013}{2011}} - \sqrt[3]{\frac{n-2011}{2013}}$$

189. (Stanford Mathematics Tournament 2012) Compute the minimum possible value of

$$(x-1)^2 + (x-2)^2 + (x-3)^2 + (x-4)^2 + (x-5)^2$$

For real values x

190. (Stanford Mathematics Tournament 2012) Find the minimum value of xy, given that

$$x^2 + y^2 + z^2 = 7$$

,

$$xz + xy + yz = 4$$

and x, y, z are real numbers

191. (TSTST 2012) Positive real numbers x, y, z satisfy $xyz + xy + yz + zx = x + y + z + 1$. Prove that

$$\frac{1}{3}\left(\sqrt{\frac{1+x^2}{1+x}} + \sqrt{\frac{1+y^2}{1+y}} + \sqrt{\frac{1+z^2}{1+z}}\right) \leq \left(\frac{x+y+z}{3}\right)^{5/8}.$$

192. (Turkey Junior MO 2012) Let a, b, c be positive real numbers satisfying $a^3 + b^3 + c^3 = a^4 + b^4 + c^4$. Show that

$$\frac{a}{a^2 + b^3 + c^3} + \frac{b}{a^3 + b^2 + c^3} + \frac{c}{a^3 + b^3 + c^2} \geq 1$$

193. (Turkey 2012) For all positive real numbers x, y, z, show that

$$\frac{x(2x-y)}{y(2z+x)} + \frac{y(2y-z)}{z(2x+y)} + \frac{z(2z-x)}{x(2y+z)} \geq 1$$

194. (Turkey TST 2012) For all positive real numbers a, b, c satisfying $ab+bc+ca \leq 1$, prove that

$$a+b+c+\sqrt{3} \geq 8abc\left(\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1}\right)$$

195. (Tuymaada 2012) Prove that for any real numbers a, b, c satisfying $abc = 1$ the following inequality holds

$$\frac{1}{2a^2+b^2+3} + \frac{1}{2b^2+c^2+3} + \frac{1}{2c^2+a^2+3} \leq \frac{1}{2}.$$

196. (USAJMO 2012) Let a, b, c be positive real numbers. Prove that

$$\frac{a^3+3b^3}{5a+b} + \frac{b^3+3c^3}{5b+c} + \frac{c^3+3a^3}{5c+a} \geq \frac{2}{3}(a^2+b^2+c^2)$$

197. (Uzbekistan 2012) Given a, b and c positive real numbers with $ab+bc+ca = 1$. Then prove that

$$\frac{a^3}{1+9b^2ac} + \frac{b^3}{1+9c^2ab} + \frac{c^3}{1+9a^2bc} \geq \frac{(a+b+c)^3}{18}$$

198. (Vietnam TST 2012) Prove that $c = 10\sqrt{24}$ is the largest constant such that if there exist positive numbers $a_1, a_2, \ldots, a_{17}$ satisfying:

$$\sum_{i=1}^{17} a_i^2 = 24, \; \sum_{i=1}^{17} a_i^3 + \sum_{i=1}^{17} a_i < c$$

then for every i, j, k such that $1 \leq 1 < j < k \leq 17$, we have that x_i, x_j, x_k are sides of a triangle.

2 Solutions

1. http://www.artofproblemsolving.com/community/c6h1084414p4785586
2. http://www.artofproblemsolving.com/community/c6h1084465p4786027
3. http://www.artofproblemsolving.com/community/c6h1084477p4786093
4. http://www.artofproblemsolving.com/community/c6h1072850p4671682
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127. http://www.artofproblemsolving.com/community/c6h564744p3301778

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