Problem 1 [Romania 1974]

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Let a, b, and c be real and positive parameters.

Solve the equation

$$\sqrt{a+bx}+\sqrt{b+cx}+\sqrt{c+ax}=\sqrt{b-ax}+\sqrt{c-bx}+\sqrt{a-cx}.$$

Çözüm: $x=0$'ın çözüm olduğu açıkçar. Dikkat edilirse, eşitliğin sağ tarafı azalan, sol tarafı artan fonksiyondur. Dolayısıyla maximum bir çözüm olabilir. O da $x=0$ dır.

Let a, b, and c be distinct nonzero real numbers such that

$$a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}.$$

Prove that $|abc|=1$.

Çözüm:

$$a-b=\frac{1}{c}-\frac{1}{b} \Rightarrow a-b=\frac{b-c}{bc}$$

$$c-a=\frac{1}{b}-\frac{1}{a} \Rightarrow c-a=\frac{a-b}{ab}$$

$$b-c=\frac{1}{a}-\frac{1}{c} \Rightarrow b-c=\frac{c-a}{ac}$$

Taraf tarafa çarpalım.

$$\cancel{(a-b)}\cdot\cancel{(c-a)}\cdot\cancel{(b-c)}=\frac{\cancel{(b-c)}}{bc}\cdot\frac{\cancel{(a-b)}}{ab}\cdot\frac{\cancel{(c-a)}}{ac}$$

$$a^2b^2c^2=1$$

$$(abc)^2=1 \Rightarrow |abc|=1.$$

Evaluate

$$\frac{3}{1!+2!+3!}+\frac{4}{2!+3!+4!}+\cdots+\frac{2001}{1999!+2000!+2001!}.$$

Çözüm:

$$\frac{k+2}{k!+(k+1)!+(k+2)!}=\frac{k+2}{k!+(k+1).k!+(k+2)(k+1).k!}$$

$$=\frac{k+2}{k!\,(1+k+1+(k+1)(k+2))}$$

$$=\frac{\cancel{k+2}}{k!.(k+2)\cancel{(k+2)}}=\frac{1}{k!.(k+2)}$$

bir ufak hile yapalım

bir hile daha

$$\frac{1}{k!(k+2)}=\frac{k+1}{(k+2)!}=\frac{(k+2)-1}{(k+2)!}=\frac{k+2}{(k+2)!}-\frac{1}{(k+2)!}$$

$$=\frac{\cancel{(k+2)}}{\cancel{(k+2)}.(k+1)!}-\frac{1}{(k+2)!}$$

$$=\frac{1}{(k+1)!}-\frac{1}{(k+2)!}$$

$k=1$ için $\quad \frac{1}{2!}-\frac{1}{3!}$

$k=2 \quad \frac{1}{3!}-\frac{1}{4!}$

$k=1999 \quad \frac{1}{2000!}-\frac{1}{2001!}$

$+$

$$\frac{1}{2}-\frac{1}{2001!}$$

Find all real numbers x for which

$$10^x + 11^x + 12^x = 13^x + 14^x.$$

Çözüm: $x=2$ yazılırsa;

$$10^2+11^2+12^2=13^2+14^2 \quad \text{sağlanır.}$$

Şimdi, her tarafı 13^x ile bölelim.

$$\left(\frac{10}{13}\right)^x+\left(\frac{11}{13}\right)^x+\left(\frac{12}{13}\right)^x=1+\left(\frac{14}{13}\right)^x$$

Dikkat edilirse, sol taraf azalan, sağ taraf artan bir fonksiyondur. Dolayısıyla, sadece $x=2$ tek sağlayan değerdir.

Prove that

$$16 < \sum_{k=1}^{80} \frac{1}{\sqrt{k}} < 17.$$

Çözüm:

$$2(\sqrt{k+1}-\sqrt{k}) = \frac{2}{\sqrt{k+1}+\sqrt{k}} < \frac{1}{\sqrt{k}}$$

$$2(\sqrt{k}-\sqrt{k-1}) = \frac{2}{\sqrt{k}+\sqrt{k-1}} > \frac{1}{\sqrt{k}}$$

$$\sum_{k=1}^{80}\frac{1}{\sqrt{k}} > 2\underbrace{\sum_{k=1}^{80}(\sqrt{k+1}-\sqrt{k})}_{8} = 16$$

$$\sum_{k=1}^{80}\frac{1}{\sqrt{k}} < 1+\underbrace{2\sum_{k=2}^{80}(\sqrt{k}-\sqrt{k-1})}_{2\sqrt{80}-2}$$

$$2\sqrt{80}-1 < 17$$

Find all complex numbers z such that

$$(3z+1)(4z+1)(6z+1)(12z+1)=2.$$

Çözüm: $8\cdot(3z+1)\cdot 6(4z+1)\cdot 4(6z+1)\cdot 2(12z+1)=2\cdot 8\cdot 6\cdot 4\cdot 2$

$(24z+8)(24z+6)(24z+4)(24z+2)=768$

$24z+5=a$ olsun.

$a^2=b$ diyelim

$(a+3)(a+1)(a-1)(a-3)=768$

$(a^2-1)(a^2-9)=768$

$a^4-10a^2+9=768$

$b^2-10b+9=768$

$b^2-10b-759=0$

$(b-33)(b+23)=0$

$b=33$ veya $b=-23$

$a=\frac{\mp\sqrt{33}-5}{24}$, $a=\frac{\mp\sqrt{23}i-5}{24}$

Let x, y, and z be positive real numbers. Prove that

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+z)(y+x)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}} \leq 1.$$

Çözüm:

$$\sqrt{(x+y).(x+z)} \geq \sqrt{xy}+\sqrt{xz}$$ (Her iki tarafın karesini alalım)

$$x^2+\cancel{xz}+\cancel{yx}+yz \geq \cancel{xy}+\cancel{xz}+2x\sqrt{yz}$$

$$x^2+yz \geq 2x\sqrt{yz}$$

aritmetik ortalama-geometrik ortalama eşitsizliğinden

$$\frac{x}{x+\sqrt{(x+y)(x+z)}} \leq \frac{x}{x+\sqrt{xy}+\sqrt{xz}} = \frac{\sqrt{x}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$$

Benzer şekilde

$$\frac{y}{y+\sqrt{(y+z)(y+x)}} \leq \frac{y}{y+\sqrt{xy}+\sqrt{zy}} = \frac{\sqrt{y}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$$

$$\frac{z}{z+\sqrt{(z+x)(z+y)}} \leq \frac{z}{z+\sqrt{xz}+\sqrt{yz}} = \frac{\sqrt{z}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$$

taraf tarafa toplarsak;

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+z)(y+x)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}} \leq \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{\sqrt{x}+\sqrt{y}+\sqrt{z}} = 1$$

Find a triple of rational numbers (a,b,c) such that

$$\sqrt[3]{\sqrt[3]{2}-1} = \sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}.$$

Çözüm: $x=\sqrt[3]{\sqrt[3]{2}-1}$ $\quad y=\sqrt[3]{2}$ diyelim.

$y^3=2$ olur; $\quad x=\sqrt[3]{y-1}$ olur.

$$1=2-1=y^3-1=(y-1)(y^2+y+1)$$

$$y^2+y+1 = \frac{3y^2+3y+3}{3} = \frac{y^3+3y^2+3y+1}{3} = \frac{(y+1)^3}{3}$$

y^3+1

bir hile yapalım

$$x^3=y-1=\frac{1}{y^2+y+1}=\frac{3}{(y+1)^3} \Rightarrow x=\frac{\sqrt[3]{y}}{y+1} \quad (1)$$

$$3=y^3+1=(y+1)(y^2-y+1)$$

$$\frac{1}{y+1}=\frac{y^2-y+1}{3} \quad (2)$$

(1) ve (2) nolu denklemlerden

$$x=\sqrt[3]{\frac{1}{9}}\left(\sqrt[3]{4}-\sqrt[3]{2}+1\right)$$

$$(a,b,c)=\left(\frac{4}{9},-\frac{2}{9},\frac{1}{9}\right)$$