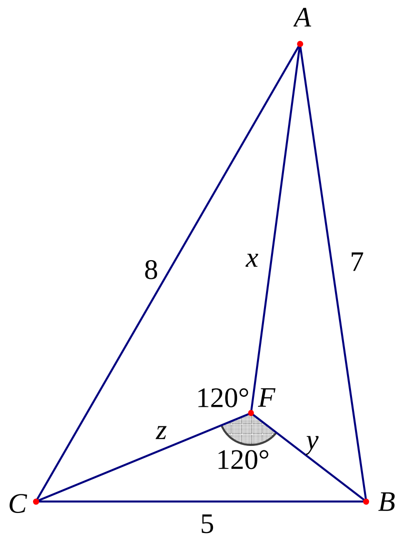
Problem: $x, y, z > 0$ real numbers satisfy the following equations:

$$y^2 + yz + z^2 = 5^2,\ z^2 + zx + x^2 = 8^2,\ x^2 + xy + y^2 = 7^2$$

Calculate value of $x + y + z$.

Solution 1: Let's draw the following ABC triangle. By cosine theorem: $y^2 + yz + z^2 = 5^2$, $z^2 + zx + x^2 = 8^2$, $x^2 + xy + y^2 = 7^2$. By Heron, $Area(ABC) = \sqrt{10 \cdot (10-5) \cdot (10-8) \cdot (10-7)}$ and $Area(ABC) = 10\sqrt{3}$.



Also, $Area(AFC) = \frac{1}{2} xz \cdot \sin 120^o$, $Area(BFC) = \frac{1}{2} yz \cdot \sin 120^o$, $Area(AFB) = \frac{1}{2} xy \cdot \sin 120^o$.

Therefore, $(xy + yz + zx)\frac{\sqrt{3}}{4} = 10\sqrt{3}$ and

$$xy + yz + zx = 40 \quad \ldots (1)$$

Now, from the equations $y^2 + yz + z^2 = 5^2$, $z^2 + zx + x^2 = 8^2$, $x^2 + xy + y^2 = 7^2$ we yields

$2(x^2 + y^2 + z^2) + (xy + yz + zx) = 138$. By (1), we get

$$x^2 + y^2 + z^2 = 49 \quad \ldots (2)$$

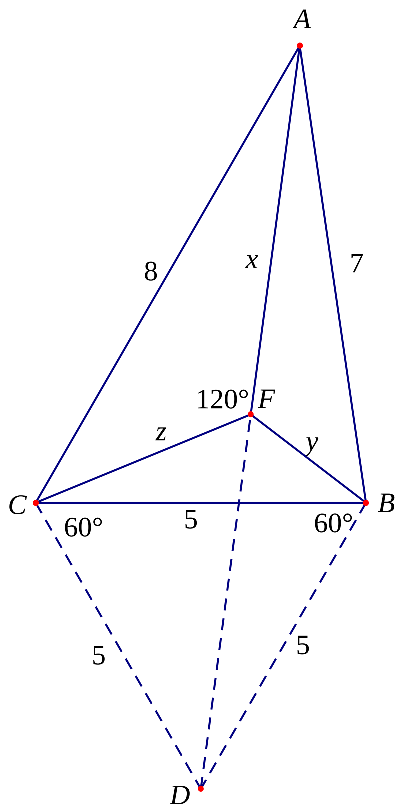
In the last step, we use perfect square identity: $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$ and $(x + y + z)^2 = 49 + 2 \cdot 40 = 129 \Rightarrow x + y + z = \sqrt{129}$.

Solution 2: *F* is Fermat point! Most popular property of this supreme point is

$$\min\{|PA|+|PB|+|PC|\}=|FA|+|FB|+|FC|=|AD|$$

Here, *P* is any point in the plane. In *ABC* triangle, we apply cosine theorem:

$$\cos(ACB)=\frac{5^2+8^2-7^2}{2\cdot 5\cdot 8}=\frac{1}{2} \text{ and } m(ACB)=60^o.$$



Therefore $m(ACD)=120^o$ and let's apply cosine theorem in the triangle *ACD*

$$|AD|^2=5^2+8^2-2\cdot 5\cdot 8\cdot\cos 120^o=129 \Rightarrow x+y+z=|AD|=\sqrt{129}$$

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