Slicing Space

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Students can be guided to make discoveries far beyond what they think they're capable of achieving. Here is a line of investigation suitable for a math club, using features of algebra, geometry and finite mathematics to generate results available in no individual undergraduate course (see [**5**]).

We consider the question: *When m-dimensional space is partitioned by n hyperplanes (i.e., subspaces of dimension* $m - 1$*), how many distinct m-dimensional subspaces are created?* We approach this first by the "sweep" method ([**1, 2**]), which has a simple, intuitive elegance, and then by a "Pascal triangle" method, which also promotes geometric intuition, but seems uninspected in the literature. Throughout this paper we assume that the n hyperplanes are in general position and that $n \geq m$.

"Sweep" approach. We begin by asking how many line segments, L_n, are formed when we place n points on a line. The obvious answer, $n + 1$, can be interpreted as the sum $1 + 1 + 1 + \cdots + 1$, where the first 1 designates the original line, and successive 1s represent the segments created as points are added.

Next, how many distinct areas, A_n, are formed when a plane is cut by n lines, as in Figure 1? As with the reasoning for a line, we begin with the plane and add successive lines. The number of areas is

$$A_n = 1 + 1 + 2 + 3 + \cdots + n = 1 + \frac{n(n+1)}{2} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2}. \quad (1)$$

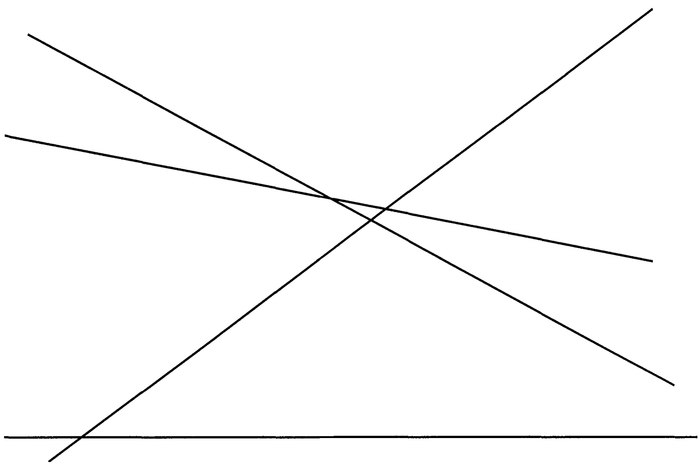

Figure 1.

In order to proceed to volumes in three-space, it will be useful to obtain (1) by a slightly different method. Again, consider the n lines in Figure 1. Since they are in general position, each line crosses every other line, producing $\binom{n}{2}$ crossings. An imaginary horizontal line drawn below all of these crossings cuts through all n lines and thus passes through $1+n$ areas. Let this imaginary line now sweep up the plane. As it passes through each of the $\binom{n}{2}$ crossings it counts one new area. Thus when it has swept up beyond all the crossings it will have passed through $1+n+\binom{n}{2}$ crossings, which is (1).

Let us continue the "sweep method" in three dimensions. Return to Figure 1, but now let each of the printed lines be the intersection of one of n planes with the imaginary plane of our page, which sits between our eyes and all the points of intersection (where three planes meet) of the n planes. The double use of Figure 1 implies that our page slices through $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}$ distinct volumes. Let our page now sweep backward. Each time our page passes through a point of intersection it slices through a further volume, but since general position implies that any three planes intersect at a point, there must be $\binom{n}{3}$ additional volumes. The total number of volumes created in three-space by n planes therefore is

$$V_n = \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}. \tag{2}$$

If we let $S_{m,n}$ be the number of m-dimensional subspaces created in m-dimensional space by the intersection of n hyperplanes, then by reasoning as above, though a bit more intricate and harder to visualize, we have

$$S_{m,n} = \sum_{i=0}^{m}\binom{n}{i}. \tag{3}$$

"Pascal triangle" approach. Let's consider Figure 1 again, but assume now that there are $n-1$ lines in the figure, each being the intersection of a plane with our page. How many new volumes will be created if our page is the nth plane? The answer is A_{n-1}, the number of areas in (1), if we replace n by $n-1$, because, as we reasoned above, each of these areas is a cross-section of a volume that is being sliced in two by the plane of our paper. Since by definition we already have V_{n-1} volumes from the previous $n-1$ planes, we have shown that

$$V_n = V_{n-1} + A_{n-1}. \tag{4}$$

Similar reasoning works even more easily in lower dimensions, producing

$$A_n = A_{n-1} + L_{n-1} \quad \text{and} \quad L_n = L_{n-1} + 1. \tag{5}$$

Using (4) and (5), we can expand V_n:

$$\begin{aligned} & V_n \\ = {} & V_{n-1} + A_{n-1} \\ = {} & V_{n-2} + A_{n-2} + A_{n-2} + L_{n-2} \\ = {} & V_{n-3} + A_{n-3} + A_{n-3} + L_{n-3} + A_{n-3} + L_{n-3} + L_{n-3} + 1. \end{aligned}$$

If we now combine like terms, taking note of how each line is formed from the previous, we have a pattern whose coefficients are those of the Pascal triangle:

$$
\begin{array}{lccccccccc}
 & & & & & V_n & & & & \\
= & & & V_{n-1} & & + & A_{n-1} & & & \\
= & & V_{n-2} & + & 2A_{n-2} & + & L_{n-2} & & & \\
= & V_{n-3} & + & 3A_{n-3} & + & 3L_{n-3} & + & 1 & & \\
= & V_{n-4} & + & 4A_{n-4} & + & 6L_{n-4} & + & 4(1) & & \\
= & V_{n-5} & + & 5A_{n-5} & + & 10L_{n-5} & + & 10(1) & & \\
= \cdots = & V_0 & + & \binom{n}{1} A_0 & + & \binom{n}{2} L_0 & + & \binom{n}{3} (1). & &
\end{array}
$$

That is, we again have (2).

To generalize to (3), we note that just as a plane divides every volume it slices into two smaller volumes, a hyperplane divides every m-dimensional region it partitions into two smaller m-dimensional regions. If x_1 through x_m are the coordinates of our m-dimensional space, then a hyperplane would have equation $a_1x_1 + a_2x_2 + \cdots + a_mx_m = k_m$. Without loss of generality, we assume that the nth hyperplane has equation $x_m = 0$ and that all points of intersection of any m of the first $n-1$ hyperplanes have $x_m > 0$. Then following the reasoning behind (4), each of the $S_{m-1,n-1}$ m-dimensional subspaces that intersect $x_m = 0$ are cut by it into two parts. Thus,

$$S_{m,n} = S_{m,n-1} + S_{m-1,n-1} \tag{6}$$

and corresponding Pascal triangles will again produce (3). (For the earliest reference, see [**6**]; for other presentations, see [**3, 4**]).

References

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Image Reconstruction in Linear Algebra

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Recently, inspired by [**1**], we have been using one and two dimensional image reconstruction problems in our introductory course in linear algebra to motivate and illustrate various topics. We suppose that real world scenes are in "black and white"

