

M483. *Proposé par Bruce Sawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Le triangle ABC est tel que $\angle BAC = 90^\circ$. Les pieds des perpendiculaires de A jusqu'aux bissectrices internes des angles $\angle ABC$ et $\angle ACB$ sont P et Q respectivement. Déterminer la mesure de $\angle PAQ$.

M484. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Résoudre l'équation

$$x^2 + 4 \left(\frac{x}{x-2} \right)^2 = 45.$$

M485. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Démontrer que

$$\prod_{k=1}^n \binom{n}{k} = \frac{1}{n!} \prod_{k=1}^n \frac{k^k}{(n-k)!}$$

pour tout $n \in \mathbb{N}$.

M486. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Combien de nombres distincts y a-t-il dans la liste

$$\frac{1^2 - 1 + 4}{1^2 + 1}, \frac{2^2 - 2 + 4}{2^2 + 1}, \frac{3^2 - 3 + 4}{3^2 + 1}, \dots, \frac{2011^2 - 2011 + 4}{2011^2 + 1} ?$$

M487. *Proposé par Samuel Gómez Moreno, Université de Jaén, Jaén, Espagne.*

Soit m un entier positif. Déterminer toutes les solutions réelles à l'équation

$$m + \sqrt{m + \sqrt{m + \cdots \sqrt{m + \sqrt{m + \sqrt{x}}}}} = x,$$

dans laquelle l'entière m a lieu n fois.

Mayhem Solutions

M440. *Proposed by the Mayhem Staff.*

In trapezoid $ABCD$, AB is parallel to DC and AD is perpendicular to AB . If $AB = 20$, $BC = 5x$, $CD = x^2 + 3x$, and $DA = 3x$, determine the value of x .

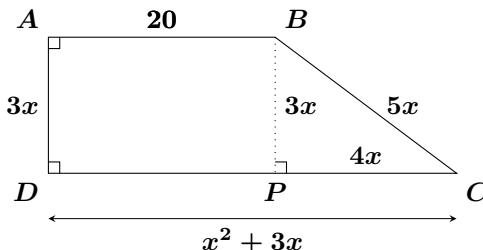
Solution by Geoffrey A. Kandall, Hamden, CT, USA.

There are two cases to consider.

Case I: $x^2 + 3x > 20$.

Let P be the foot of the perpendicular from B to DC . Then $BP = 3x$ and, according to Pythagoras, $PC = 4x$. Therefore,

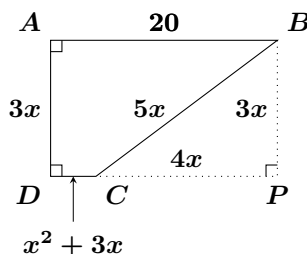
$$\begin{aligned}x^2 + 3x &= 20 + 4x, \\x^2 - x - 20 &= 0, \\(x - 5)(x + 4) &= 0, \\x &= 5 \quad (\text{since } x > 0).\end{aligned}$$



Case II: $x^2 + 3x < 20$.

Proceeding as in Case I, we obtain

$$\begin{aligned}(x^2 + 3x) + 4x &= 20, \\x^2 + 7x - 20 &= 0, \\x &= \frac{-7 + \sqrt{129}}{2}.\end{aligned}$$



Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SCOTT BROWN, Auburn University, Montgomery, AL, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ANTONIO LEDESMA LÓPEZ, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania. Eight incorrect solutions were received. Most of the incorrect solutions neglected one of the cases.

M442. *Proposed by Carl Libis, Cumberland University, Lebanon, TN, USA.*

Consider the square array

$$\begin{bmatrix} 1 & 2 & \cdots & n-1 & n \\ n+1 & n+2 & \cdots & 2n-1 & 2n \\ \vdots & \vdots & & \vdots & \vdots \\ (n-1)n+1 & (n-1)n+2 & \cdots & n^2-1 & n^2 \end{bmatrix}$$

formed by listing the numbers 1 to n^2 in order in consecutive rows. Determine the sum of the numbers on each diagonal. How does this sum compare to the "magic constant" that would be obtained if the n^2 entries were rearranged to form a magic square?

Solución de Ricard Peiró, IES “Abastos”, Valencia, Spain.

Los elementos de la diagonal principal son: $1, n + 2, 2n + 3, \dots, (n - 1)n + n$. La suma es:

$$\begin{aligned} D_1(n) &= 1 + (n + 2) + (2n + 3) + \dots + [(n - 1)n + n] \\ &= [1 + 2 + 3 + \dots + n] + n[1 + 2 + 3 + \dots + (n - 1)] \\ &= \frac{n(n + 1)}{2} + n \left(\frac{(n - 1)n}{2} \right) \\ &= \frac{n^3 + n}{2}. \end{aligned}$$

Los elementos de la diagonal secundaria son: $n, 2n - 1, 3n - 2, \dots, n \cdot n - (n - 1)$. La suma es:

$$\begin{aligned} D_2(n) &= n + (2n - 1) + (3n - 2) + \dots + [n \cdot n - (n - 1)] \\ &= n[1 + 2 + 3 + \dots + n] - [1 + 2 + 3 + \dots + (n - 1)] \\ &= n \left(\frac{n(n + 1)}{2} \right) - \frac{(n - 1)n}{2} \\ &= \frac{n^3 + n}{2}. \end{aligned}$$

La constante mágica de un cuadrado mágico $n \times n$ es:

$$\begin{aligned} M(n) &= \frac{1}{n}(1 + 2 + 3 + \dots + n^2) \\ &= \frac{1}{n} \left(\frac{n^2(n^2 + 1)}{2} \right) \\ &= \frac{n^3 + n}{2} = D_1(n) = D_2(n). \end{aligned}$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; JACLYN CHANG, student, University of Calgary, Calgary, AB; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ANTONIO LEDESMA LÓPEZ, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; and ALLEN ZHU, Conestoga High School, Berwyn, PA, USA.

M445. *Proposed by the Mayhem Staff.*

The lines with equations $y = x + 1$, $y = mx - 1$, and $y = -4x + 2m$ pass through the same point. Determine all possible values for m .

Solution by Afiffah Nur Mila Husniana, student, SMPN 8, Yogyakarta, Indonesia.

Given are three linear equations

$$y = x + 1, \tag{1}$$

$$y = mx - 1, \tag{2}$$

$$y = -4x + 2m. \tag{3}$$

From (1) and (2) we have

$$\begin{aligned}x + 1 &= mx - 1 \\x(1 - m) &= -2 \\x &= \frac{-2}{1 - m}\end{aligned}\tag{4}$$

From (2) and (3) we have

$$\begin{aligned}mx - 1 &= -4x + 2m \\x(m + 4) &= 2m + 1 \\x &= \frac{2m + 1}{m + 4}\end{aligned}\tag{5}$$

From (4) and (5) we have

$$\begin{aligned}\frac{-2}{1 - m} &= \frac{2m + 1}{m + 4} \\-2(m + 4) &= (1 - m)(2m + 1) \\-2m - 8 &= 2m - 2m^2 + 1 - m \\2m^2 - 3m - 9 &= 0 \\(2m + 3)(m - 3) &= 0\end{aligned}$$

So the possible values for m are $m = -\frac{3}{2}$ or $m = 3$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; DAVINIA CERVERA GARCÍA, Club Mathématique de l'Institut de Ecuación Secundaria No. 1, Requena-Valencia, Spain; MUHAMMAD HAFIZ FARIZI, student, SMPN 8, Yogyakarta, Indonesia; G.C. GREUBEL, Newport News, VA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; DEBRA A. OHL, student, Angelo State University, San Angelo, TX, USA; KONSTANTINOS AL. NAKOS, Agrinio, Greece; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. Two incorrect solutions were submitted.

M446. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Let a , b , and c be positive digits. Suppose that b equals the product of a , b , and c , and $\underline{ac} = a + b + c$. Determine a , b , and c . (Here \underline{ab} is the two-digit positive integer with tens digit a and units digit b .)

Solution by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA.

We are given

$$\begin{aligned}b &= a \cdot b \cdot c \\10a + c &= a + b + c\end{aligned}$$

Since a, b, c are positive digits, then $1 \leq a, b, c \leq 9$. Since $b \neq 0$ then $b = a \cdot b \cdot c$ gives $a \cdot c = 1$; which implies that $a = 1 = c$. From $10a + c = a + b + c$, then we have $10 \cdot 1 + 1 = 1 + b + 1$; hence $b = 11 - 2 = 9$. Therefore $a = 1, b = 9, c = 1$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; AFIFFAH NUUR MILA HUSNIANA, student, SMPN 8, Yogyakarta, Indonesia; YOUNGHUAN JUNG, The Woodlands School, Mississauga, ON; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; DONGCHAN LEE, University of Toronto, Toronto, ON; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; MUHAMMAD ROIHAN MUNAJIH, student, SMPN 8, Yogyakarta, Indonesia; ALEECE NALBANDIAN, California State University, Fresno, CA, USA; DEBRA A. OHL, student, Angelo State University, San Angelo, TX, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; ANDRÉS PLANELLÉS CÁRCEL, Club Matemàtic de l'Institut de Ecuación Secundaria No. 1, Requena-Valencia, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; and INGESTI BILKIS ZULFATINAAS, student, SMPN 8, Yogyakarta, Indonesia. One incorrect solution was submitted.

M447. Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.

Let $ABCD$ be a parallelogram. The sides AB and AD are extended to points E and F (respectively) so that E, C , and F all lie on a straight line. Prove that $BE \cdot DF = AB \cdot AD$.

Solution by George Apostolopoulos, Messolonghi, Greece.

The triangles BCE and FDC are similar, so $\frac{BE}{DC} = \frac{BC}{DF}$. Since $ABCD$ is a parallelogram we know $BC = AD$ and $DC = AB$. So $\frac{BE}{AB} = \frac{AD}{DF}$, hence $BE \cdot DF = AB \cdot AD$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; AFIFFAH NUUR MILA HUSNIANA, student, SMPN 8, Yogyakarta, Indonesia; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; DONGCHAN LEE, University of Toronto, Toronto, ON; DEBRA A. OHL, student, Angelo State University, San Angelo, TX, USA; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; JORGE SEVILLA LACRUZ, Club Matemàtic de l'Institut de Ecuación Secundaria No. 1, Requena-Valencia, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; LOU VANG, California State University, Fresno, CA, USA; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and INGESTI BILKIS ZULFATINAAS, student, SMPN 8, Yogyakarta, Indonesia. One incorrect solution was submitted.

M448. Proposed by the Mayhem Staff.

A polyhedron with exactly $m + n$ faces has m faces that are quadrilaterals and n faces that are triangles. Exactly four faces meet at each vertex. Prove that $n = 8$.

Solution by Dongchan Lee, University of Toronto, Toronto, ON.

The number of faces is $F = m + n$. Since there are 4 vertices and 4 edges in a quadrilateral, and 3 vertices and 3 edges in a triangle, then the total number of edges would be $E = \frac{4m+3n}{2}$. The total number of vertices will be $V = \frac{4m+3n}{4}$ since it is given that exactly four faces meet at each vertex. Using Euler's polyhedron formula, which says that the sum of the number of faces and the number of vertices is equal to the number of edges plus two,

$$\begin{aligned} F + V &= E + 2, \\ m + n + \frac{4m + 3n}{4} &= \frac{4m + 3n}{2} + 2. \end{aligned}$$

Solving the equation, we get $n = 8$.

Also solved by ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; JORGE ARMERO JIMÉNEZ, Club Mathématique de l'Institut de Ecuación Secundaria No. 1, Requena-Valencia, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania.

M449. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

$$\text{Let } E(x) = \frac{4^x}{4^x + 2}.$$

(a) Prove that $E(x) + E(1 - x) = 1$.

(b) Find the value of $E\left(\frac{1}{2010}\right) + E\left(\frac{2}{2010}\right) + \cdots + E\left(\frac{2008}{2010}\right) + E\left(\frac{2009}{2010}\right)$.

Solution by Afiffah Nuur Mila Husniana, student, SMPN 8, Yogyakarta, Indonesia.

(a) From the given equation $E(x) = \frac{4^x}{4^x + 2}$ so,

$$\begin{aligned} E(x) + E(1 - x) &= \frac{4^x}{4^x + 2} + \frac{4^{1-x}}{4^{1-x} + 2} \\ &= \frac{4^x(4^{1-x} + 2) + 4^{1-x}(4^x + 2)}{(4^x + 2)(4^{1-x} + 2)} \\ &= \frac{4 + 2(4^x) + 4 + 2(4^{1-x})}{4 + 2(4^x) + 2(4^{1-x}) + 4} \\ &= \frac{8 + 2(4^x) + 2(4^{1-x})}{8 + 2(4^x) + 2(4^{1-x})} \\ &= 1 \end{aligned}$$

[Ed. - Note that $E(1-x) = \frac{4^{1-x}}{4^{1-x} + 2} \times \frac{\frac{4^x}{2}}{\frac{4^x}{2}} = \frac{2}{2 + 4^x}$ and the conclusion follows immediately.]

(b) From (a) we know that $E(x) + E(1 - x) = 1$, thus

$$E\left(\frac{1}{2010}\right) + E\left(\frac{2009}{2010}\right) = E\left(\frac{1}{2010}\right) + E\left(1 - \frac{1}{2010}\right) = 1$$

$$E\left(\frac{2}{2010}\right) + E\left(\frac{2008}{2010}\right) = E\left(\frac{2}{2010}\right) + E\left(1 - \frac{2}{2010}\right) = 1$$

and so on. Then we have

$$E\left(\frac{1}{2010}\right) + E\left(\frac{2}{2010}\right) + \dots + E\left(\frac{2008}{2010}\right) + E\left(\frac{2009}{2010}\right) = 1004 \times 1 + E\left(\frac{1005}{2010}\right)$$

Since $E\left(\frac{1005}{2010}\right) = \frac{4^{\frac{1}{2}}}{4^{\frac{1}{2}} + 2}$, then $E\left(\frac{1005}{2010}\right) = \frac{1}{2}$.

Therefore the sum is **1004.5**.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ALPER CAY and LOKMAN GOKCE, Geomania Problem Group, Kayseri, Turkey; CHAO-PING CHEN, Henan Polytechnic University, Jiaozuo City, China and Mihály Bencze, Lajos Aprily High-school, Brasov, Romania; DIANA DOMINGUEZ, California State University, Fresno, CA, USA; MUHAMMAD HAFIZ FARIZI, student, SMPN 8, Yogyakarta, Indonesia; PABLO PARDAL GARCÉS, Club Mathématique de l'Institut de Ecuación Secundaria No. 1, Requena-Valencia, Spain; G.C. GREUBEL, Newport News, VA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; DONGCHAN LEE, University of Toronto, Toronto, ON; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; KONSTANTINOS AL. NAKOS, Agrinio, Greece; CARLOS TORRES NINAHUANCA, Lima, Perú; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and INGESTI BILKIS ZULFATINAAS, student, SMPN 8, Yogyakarta, Indonesia;

M450. Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

Prove that if n is an odd positive integer, then $n^{n+2} + (n+2)^n$ is divisible by $2(n+1)$.

Solution by Osman Ekiz, Eskisehir, Turkey.

From the Binomial Theorem, $(n+2)^n$ can be written as:

$$\begin{aligned} (n+2)^n &= ((n+1)+1)^n \\ &= \binom{n}{0}(n+1)^n + \binom{n}{1}(n+1)^{n-1} + \dots + \binom{n}{n-1}(n+1) + 1. \end{aligned}$$

Since n is an odd number we can also write:

$$\begin{aligned} n^{n+2} &= (n+1-1)^{n+2} \\ &= \binom{n+2}{0}(n+1)^{n+2} - \binom{n+2}{1}(n+1)^{n+1} + \dots \\ &\quad + \binom{n+2}{n+1}(n+1) - 1. \end{aligned}$$

If we add the two expansions together, the constant terms cancel each other. Therefore, we have:

$$\begin{aligned} (n+2)^n + n^{n+2} = & \left[\binom{n}{0} (n+1)^n + \binom{n}{1} (n+1)^{n-1} + \dots \right. \\ & + \binom{n}{n-1} (n+1) + \binom{n+2}{0} (n+1)^{n+2} \\ & \left. - \binom{n+2}{1} (n+1)^{n+1} + \dots + \binom{n+2}{n+1} (n+1) \right] \end{aligned}$$

Since all of the terms have a factor of $n+1$, then $(n+2)^n + n^{n+2}$ is divisible by $n+1$ and we can rewrite the expression as:

$$\begin{aligned} (n+2)^n + n^{n+2} = (n+1) & \left[\binom{n}{0} (n+1)^{n-1} + \binom{n}{1} (n+1)^{n-2} + \dots \right. \\ & + \binom{n}{n-1} + \binom{n+2}{0} (n+1)^{n+1} \\ & \left. - \binom{n+2}{1} (n+1)^n + \dots + \binom{n+2}{n+1} \right] \end{aligned}$$

Now we must prove that the expression in the square brackets above is an even number. Since we know that $n+1$ is an even number, all of the terms with a factor of $n+1$ are also even. Then we are left with only two terms, $\binom{n}{n-1}$ and $\binom{n+2}{n+1}$. Since $\binom{n}{n-1} = n$ and $\binom{n+2}{n+1} = n+2$, then we have $\binom{n}{n-1} + \binom{n+2}{n+1} = 2n+2$ which is an even number.

Hence $(n+2)^n + n^{n+2}$ is divisible by $2(n+1)$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ADAM GREGSON, teacher, University of Toronto Schools, Toronto, ON; G.C. GREUBEL, Newport News, VA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ANTONIO LEDESMA LÓPEZ, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain; DONGCHAN LEE, University of Toronto, Toronto, ON; SALLY LI, student, Marc Garneau Collegiate Institute, Toronto, ON; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.

Note that since n is odd $\frac{n^{n+2} + 1}{n+1} = n^{n+1} - n^n + n^{n-1} - \dots + 1$ is odd. Also $\frac{(n+2)^n - 1}{n+1} = (n+2)^{n-1} + (n+2)^n + \dots + 1$ is also odd. Thus their sum is even.